Lecture 3: Loss Functions and Optimization
Administrative: Assignment 1

Released last week, due **Wed 4/22 at 11:59pm**
Administrative: Project proposal

Due **Wed 4/27**

TA expertise listed on piazza

There is a Piazza thread to find teammates

Slack has topic specific channels for you to use
Administrative: Midterm

24-Hours open notes exam

Combination of True/False, Multiple Choice, Short Answer, Coding

Will be released May 12th, specific time TBD

Due May 13th, 24 hours from release time

The exam should take 3-4 hours to finish
Administrative: Midterm Updates

University has updated guidance on administering exams in spring quarter. In order to comply with the current policies, we have changed the exam format as the following to be consistent with exams in previous offerings of cs 231n:

**Date:** released on Tuesday 5/12 (open for 24 hours to choose 1hr 40 mins time frame)

**Format:** Timestamped with Gradescope
Administrative: Piazza

Please make sure to check and read all pinned piazza posts.
Image Classification: A core task in Computer Vision

(assume given a set of labels)
{dog, cat, truck, plane, ...}

This image by Nikita is licensed under CC-BY 2.0
Recall from last time: Challenges of recognition

- Viewpoint
- Illumination
- Deformation
- Occlusion
- Clutter
- Intraclass Variation
Recall from last time: data-driven approach, kNN

airplane  automobile  bird  cat  deer  dog  frog  horse  ship  truck

train  test

train  validation  test

1-NN classifier  5-NN classifier
Recall from last time: Linear Classifier

\[ f(x, W) = Wx + b \]
Interpreting a Linear Classifier: Visual Viewpoint

airplane  automobile  bird  cat  deer  dog  frog  horse  ship  truck

Input image

W

0.2  -0.5
0.1  2.0
1.5  1.3
2.1  0.0
0  0.25
0.2  -0.3

b

1.1
3.2
-1.2

Score

-96.8
437.9
61.95

plane  car  bird  cat  deer  dog  frog  horse  ship  truck
Example with an image with 4 pixels, and 3 classes (cat/dog/ship)

Algebraic Viewpoint

\[ f(x, W) = Wx \]

Visual Viewpoint

Input image

\[
\begin{bmatrix}
0.2 & -0.5 & 0.1 & 2.0 \\
1.5 & 1.3 & 2.1 & 0.0 \\
0 & 0.25 & 0.2 & -0.3 \\
2 & 24 & 231 & 56
\end{bmatrix}
\]

\[
\begin{bmatrix}
1.1 \\
3.2 \\
-1.2
\end{bmatrix}
\]

Score

\[
\begin{bmatrix}
-96.8 \\
437.9 \\
61.95
\end{bmatrix}
\]

W

\[
\begin{bmatrix}
0.2 & -0.5 \\
0.1 & 2.0 \\
1.5 & 1.3 \\
0.2 & -0.3
\end{bmatrix}
\]

\[
\begin{bmatrix}
1.1 \\
3.2 \\
-1.2
\end{bmatrix}
\]

b

\[
\begin{bmatrix}
-96.8 \\
437.9 \\
61.95
\end{bmatrix}
\]
Interpreting a Linear Classifier: Geometric Viewpoint

\[ f(x, W) = Wx + b \]

Array of \(32 \times 32 \times 3\) numbers (3072 numbers total)
Recall from last time: Linear Classifier

1. Define a **loss function** that quantifies our unhappiness with the scores across the training data.

2. Come up with a way of efficiently finding the parameters that minimize the loss function. *(optimization)*
Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x, W) = WX$ are:

<table>
<thead>
<tr>
<th></th>
<th>cat</th>
<th>car</th>
<th>frog</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2</td>
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<td></td>
</tr>
<tr>
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<tbody>
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A **loss function** tells how good our current classifier is.

Given a dataset of examples

$$\{(x_i, y_i)\}_{i=1}^{N}$$

Where $x_i$ is image and $y_i$ is (integer) label.
Suppose: 3 training examples, 3 classes. With some $W$ the scores $f(x, W) = Wx$ are:

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A **loss function** tells how good our current classifier is.

Given a dataset of examples

\[
\{(x_i, y_i)\}_{i=1}^{N}
\]

Where $x_i$ is image and $y_i$ is (integer) label.

Loss over the dataset is a average of loss over examples:

\[
L = \frac{1}{N} \sum_{i} L_i(f(x_i, W), y_i)
\]
Suppose: 3 training examples, 3 classes. With some $W$ the scores $f(x, W) = Wx$ are:

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**Multiclass SVM loss:**

Given an example $(x_i, y_i)$ where $x_i$ is the image and $y_i$ is the (integer) label, and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \begin{cases} 0 & \text{if } s_{y_i} \geq s_j + 1 \\ s_j - s_{y_i} + 1 & \text{otherwise} \end{cases}$$

$$= \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$
Suppose: 3 training examples, 3 classes. With some $W$ the scores $f(x, W) = Wx$ are:

cat: 3.2, 1.3, 2.2

car: 5.1, 4.9, 2.5

frog: -1.7, 2.0, -3.1

Interpreting Multiclass SVM loss:

$$L_i = \sum_{j \neq y_i} \begin{cases} 
0 & \text{if } s_{y_i} \geq s_j + 1 \\
(s_j - s_{y_i}) + 1 & \text{otherwise}
\end{cases} = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$
Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x, W) = Wx$ are:

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<tbody>
<tr>
<td>score</td>
<td>3.2</td>
<td>5.1</td>
<td>-1.7</td>
</tr>
<tr>
<td></td>
<td>1.3</td>
<td>4.9</td>
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Interpreting Multiclass SVM loss:

Given an example where $x$ is the image and $y_i$ is the (integer) label, and using the shorthand for the scores vector:

The SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \begin{cases} 0 & \text{if } s_{y_i} \geq s_j + 1 \\ s_j - s_{y_i} + 1 & \text{otherwise} \end{cases}$$

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Suppose: 3 training examples, 3 classes. With some $W$ the scores $f(x, W) = Wx$ are:

<table>
<thead>
<tr>
<th>Class</th>
<th>Score $s_y$</th>
<th>Score $s_j$ (other classes)</th>
<th>Margin</th>
</tr>
</thead>
<tbody>
<tr>
<td>cat</td>
<td>3.2</td>
<td>1.3, 2.2</td>
<td></td>
</tr>
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Interpreting Multiclass SVM loss:

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(s_j - s_{y_i}) + 1 & \text{otherwise}
\end{cases}$$

$$= \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

“Hinge loss”
Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x, W) = WX$ are:

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Multiclass SVM loss:

Given an example $(x_i, y_i)$ where $x_i$ is the image and where $y_i$ is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$
Suppose: 3 training examples, 3 classes. With some $W$ the scores $f(x, W) = Wx$ are:

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**Multiclass SVM loss:**

Given an example $(x_i, y_i)$ where $x_i$ is the image and $y_i$ is the (integer) label, and using the shorthand for the scores vector: $s = f(x_i, W)$, the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$= \max(0, 5.1 - 3.2 + 1) + \max(0, -1.7 - 3.2 + 1)$$

$$= \max(0, 2.9) + \max(0, -3.9)$$

$$= 2.9 + 0$$

$$= 2.9$$
Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x, W) = Wx$ are:

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<td></td>
<td></td>
</tr>
<tr>
<td>Losses</td>
<td>2.9</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>2.9</td>
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**Multiclass SVM loss:**

Given an example $(x_i, y_i)$ where $x_i$ is the image and $y_i$ is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$= \max(0, 1.3 - 4.9 + 1) + \max(0, 2.0 - 4.9 + 1)$$

$$= \max(0, -2.6) + \max(0, -1.9)$$

$$= 0 + 0$$

$$= 0$$
Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x, W) = Wx$ are:

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Losses: 2.9 0 12.9

Multiclass SVM loss:

Given an example $(x_i, y_i)$ where $x_i$ is the image and $y_i$ is the (integer) label, and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$= \max(0, 2.2 - (-3.1) + 1) + \max(0, 2.5 - (-3.1) + 1)$$

$$= \max(0, 6.3) + \max(0, 6.6)$$

$$= 6.3 + 6.6$$

$$= 12.9$$
Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x, W) = Wx$ are:

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Losses: 2.9 0 12.9

**Multiclass SVM loss:**

Given an example $(x_i, y_i)$ where $x_i$ is the image and $y_i$ is the (integer) label,

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the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Loss over full dataset is average:

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i$$

$$L = \frac{(2.9 + 0 + 12.9)}{3} = 5.27$$
Suppose: 3 training examples, 3 classes. With some $W$ the scores $f(x, W) = Wx$ are:

<table>
<thead>
<tr>
<th>Class</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>cat</td>
<td>1.3</td>
</tr>
<tr>
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<td>4.9</td>
</tr>
<tr>
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</table>

Losses: 0

Multiclass SVM loss:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q1: What happens to loss if car scores decrease by 0.5 for this training example?

Q2: What is the min/max possible SVM loss $L_i$?

Q3: At initialization $W$ is small so all $s \approx 0$. What is the loss $L_i$, assuming $N$ examples and $C$ classes?
Suppose: 3 training examples, 3 classes. With some $W$ the scores $f(x, W) = Wx$ are:

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### Multiclass SVM loss:

Given an example $(x_i, y_i)$ where $x_i$ is the image and where $y_i$ is the (integer) label, and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q4: What if the sum was over all classes? (including $j = y_i$)
Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x, W) = Wx$ are:

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**Multiclass SVM loss:**

Given an example $(x_i, y_i)$ where $x_i$ is the image and $y_i$ is the (integer) label, and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Q5: What if we used mean instead of sum?
Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x, W) = Wx$ are:

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<tr>
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### Multiclass SVM loss:

Given an example $(x_i, y_i)$ where $x_i$ is the image and $y_i$ is the (integer) label, and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

**Q6:** What if we used

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)^2$$
Multiclass SVM Loss: Example code

\[ L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \]

```python
def L_i_vectorized(x, y, W):
    scores = W.dot(x)
    margins = np.maximum(0, scores - scores[y] + 1)
    margins[y] = 0
    loss_i = np.sum(margins)
    return loss_i
```

# First calculate scores
# Then calculate the margins \( s_j - s_{y_i} + 1 \)
# only sum \( j \) is not \( y_i \), so when \( j = y_i \) set to zero.
# sum across all \( j \)
\[ f(x, W) = Wx \]

\[ L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1) \]

Q7. Suppose that we found a \( W \) such that \( L = 0 \). Is this \( W \) unique?
\[ f(x, W) = Wx \]

\[
L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)
\]

E.g. Suppose that we found a \( W \) such that \( L = 0 \). Is this \( W \) unique?

No! \( 2W \) is also has \( L = 0! \)
Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x, W) = Wx$ are:

<table>
<thead>
<tr>
<th></th>
<th>cat</th>
<th>frog</th>
</tr>
</thead>
<tbody>
<tr>
<td>score</td>
<td>3.2</td>
<td>-1.7</td>
</tr>
<tr>
<td>score</td>
<td>1.3</td>
<td>2.0</td>
</tr>
<tr>
<td>score</td>
<td>2.2</td>
<td>-3.1</td>
</tr>
</tbody>
</table>

Losses: 2.9

= $\frac{1}{2}$ \sum_{i=1}^{3} \max(0, s_j - s_i + 1)$

Before:

= max(0, 1.3 - 4.9 + 1) + max(0, 2.0 - 4.9 + 1)
= max(0, -2.6) + max(0, -1.9)
= 0 + 0
= 0

With $W$ twice as large:

= max(0, 2.6 - 9.8 + 1) + max(0, 4.0 - 9.8 + 1)
= max(0, -6.2) + max(0, -4.8)
= 0 + 0
= 0
\[ f(x, W) = Wx \]

\[ L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1) \]

E.g. Suppose that we found a \( W \) such that \( L = 0 \). Is this \( W \) unique?

No! \( 2W \) is also has \( L = 0 \)!
How do we choose between \( W \) and \( 2W \)?
Regularization

\[ L(W) = \frac{1}{N} \sum_{i=1}^{N} L_i(f(x_i, W), y_i) \]

**Data loss**: Model predictions should match training data.
Regularization

\[ L(W) = \frac{1}{N} \sum_{i=1}^{N} L_i(f(x_i, W), y_i) + \lambda R(W) \]

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing too well on training data
Regularization intuition: toy example training data
Regularization intuition: Prefer Simpler Models
Regularization: Prefer Simpler Models

Regularization pushes against fitting the data *too* well so we don’t fit noise in the data.
Regularization

\[ L(W) = \frac{1}{N} \sum_{i=1}^{N} L_i(f(x_i, W), y_i) + \lambda R(W) \]

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing too well on training data

Occam’s Razor: Among multiple competing hypotheses, the simplest is the best,
William of Ockham 1285-1347
Regularization

\[ L(W) = \frac{1}{N} \sum_{i=1}^{N} L_i(f(x_i, W), y_i) + \lambda R(W) \]

- **Data loss**: Model predictions should match training data
- **Regularization**: Prevent the model from doing too well on training data

\[ \lambda = \text{regularization strength (hyperparameter)} \]
Regularization

$$L(W) = \frac{1}{N} \sum_{i=1}^{N} L_i(f(x_i, W), y_i) + \lambda R(W)$$

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing too well on training data

$\lambda = \text{regularization strength (hyperparameter)}$

Simple examples

**L2 regularization:** $R(W) = \sum_k \sum_l W_{k,l}^2$

**L1 regularization:** $R(W) = \sum_k \sum_l |W_{k,l}|$

**Elastic net (L1 + L2):** $R(W) = \sum_k \sum_l \beta W_{k,l}^2 + |W_{k,l}|$
Regularization

\[ L(W) = \frac{1}{N} \sum_{i=1}^{N} L_i(f(x_i, W), y_i) + \lambda R(W) \]

\( \lambda \) = regularization strength (hyperparameter)

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing too well on training data

Simple examples

L2 regularization: \( R(W) = \sum_k \sum_l W_{k,l}^2 \)

L1 regularization: \( R(W) = \sum_k \sum_l |W_{k,l}| \)

Elastic net (L1 + L2): \( R(W) = \sum_k \sum_l \beta W_{k,l}^2 + |W_{k,l}| \)

More complex:

Dropout
Batch normalization
Stochastic depth, fractional pooling, etc
Regularization

\[ L(W) = \frac{1}{N} \sum_{i=1}^{N} L_i(f(x_i, W), y_i) + \lambda R(W) \]

\( \lambda \) = regularization strength (hyperparameter)

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing too well on training data

Why regularize?
- Express preferences over weights
- Make the model simple so it works on test data
- Improve optimization by adding curvature
Regularization: Expressing Preferences

Which of \( w_1 \) or \( w_2 \) will the L2 regularizer prefer?

\[
x = [1, 1, 1, 1]
\]

\[
w_1 = [1, 0, 0, 0]
\]

\[
w_2 = [0.25, 0.25, 0.25, 0.25]
\]

\[
w_1^T x = w_2^T x = 1
\]

L2 Regularization

\[
R(W) = \sum_k \sum_l W_{k,l}^2
\]
Regularization: Expressing Preferences

L2 Regularization

\[ R(W) = \sum_k \sum_l W_{k,l}^2 \]

Which of \( w_1 \) or \( w_2 \) will the L2 regularizer prefer?

L2 regularization likes to “spread out” the weights

\[ x = [1, 1, 1, 1] \]
\[ w_1 = [1, 0, 0, 0] \]
\[ w_2 = [0.25, 0.25, 0.25, 0.25] \]

\[ w_1^T x = w_2^T x = 1 \]
Regularization: Expressing Preferences

\[ x = [1, 1, 1, 1] \]
\[ w_1 = [1, 0, 0, 0] \]
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\[
w_1^T x = w_2^T x = 1
\]

L2 Regularization
\[
R(W) = \sum_k \sum_l W_{k,l}^2
\]

Which of \( w_1 \) or \( w_2 \) will the L2 regularizer prefer?
L2 regularization likes to “spread out” the weights

Which one would L1 regularization prefer?
Softmax classifier
Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities

cat 3.2

frog -1.7
Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities

\[ s = f(x_i; W) \]

\[ P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \]

Softmax Function

<table>
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<th>Class</th>
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<tbody>
<tr>
<td>cat</td>
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Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities

\[
s = f(x_i; W)
\]

\[
P(Y = k|X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \quad \text{Softmax Function}
\]

Probabilities must be \( \geq 0 \)

- Cat: 3.2 \rightarrow 24.5
- Car: 5.1 \rightarrow 164.0
- Frog: -1.7 \rightarrow 0.18

unnormalized probabilities
Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities

\[ s = f(x_i; W) \]

\[ P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \]

Softmax Function

- Probabilities must be \( \geq 0 \)
- Probabilities must sum to 1

<table>
<thead>
<tr>
<th>Class</th>
<th>Score</th>
<th>Unnormalized Probabilities</th>
<th>Normalized Probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>cat</td>
<td>3.2</td>
<td>24.5</td>
<td>0.13</td>
</tr>
<tr>
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Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities

\[ s = f(x_i; W) \]

\[ P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \]

Probabilities must be >= 0

Probabilities must sum to 1

Unnormalized log-probabilities / logits

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Unnormalized probabilities

probabilities
Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities

$s = f(x_i; W)$

$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$

Probabilities must be $\geq 0$

Probabilities must sum to 1

$L_i = -\log P(Y = y_i | X = x_i)$

unnormalized

log-probabilities / logits

unnormalized probabilities

probabilities

Unnormalized log-probabilities / logits

exp

normalize

$\exp 3.2$

$\exp 5.1$

$\exp -1.7$

$24.5$

$164.0$

$0.18$

$0.13$

$0.87$

$0.00$

$\rightarrow L_i = -\log(0.13) = 2.04$
Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities

\[ s = f(x_i; W) \]

\[ P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \]

Probabilities must be >= 0
Probabilities must sum to 1

Unnormalized log-probabilities / logits

Cat: 3.2
Car: 5.1
Frog: -1.7

Unnormalized probabilities

Cat: 24.5
Car: 164.0
Frog: 0.18

Probabilities

Cat: 0.13
Car: 0.87
Frog: 0.00

\[ L_i = -\log P(Y = y_i | X = x_i) \]

\[ \rightarrow L_i = -\log(0.13) = 2.04 \]

Maximum Likelihood Estimation
Choose weights to maximize the likelihood of the observed data
(See CS 229 for details)
Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities:

\[ s = f(x_i; W) \]

\[ P(Y = k|X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \]

Probabilities must be \( \geq 0 \)

Probabilities must sum to 1

Unnormalized log-probabilities / logits

Unnormalized probabilities

probabilities

Correct probs

\[ L_i = -\log P(Y = y_i|X = x_i) \]

exp

normalize

compare
Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities

\[ s = f(x_i; W) \]

\[ P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \]

Softmax Function

Want to interpret raw classifier scores as probabilities

Probabilities must be \( \geq 0 \)

Probabilities must sum to 1

Unnormalized log-probabilities / logits

Unnormalized probabilities

Correct probs

\[ L_i = - \log P(Y = y_i | X = x_i) \]

Kullback–Leibler divergence

\[ D_{KL}(P || Q) = \sum_y P(y) \log \frac{P(y)}{Q(y)} \]
### Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

\[
s = f(x_i; W)
\]

\[
P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}
\]

#### Softmax Function

<table>
<thead>
<tr>
<th>Category</th>
<th>Value</th>
<th>Unnormalized log-probabilities / logits</th>
<th>Probabilities</th>
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<td></td>
<td>0.00</td>
</tr>
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</table>

Probabilities must be \(\geq 0\)

Probabilities must sum to 1

Cross Entropy

\[
H(P, Q) = H(p) + D_{KL}(P || Q)
\]

Correct probs

\[
L_i = -\log P(Y = y_i | X = x_i)
\]

Fei-Fei Li, Ranjay Krishna, Danfei Xu
Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

\[ s = f(x_i; W) \quad P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \]

Maximize probability of correct class

\[ L_i = - \log P(Y = y_i | X = x_i) \]

Putting it all together:

\[ L_i = - \log \left( \frac{e^{s_{y_i}}}{\sum_j e^{s_j}} \right) \]
Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities

\[ s = f(x_i; W) \]

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\[ L_i = -\log P(Y = y_i | X = x_i) \]

\[ L_i = -\log \left( \frac{e^{s_{y_i}}}{\sum_j e^{s_j}} \right) \]

Q1: What is the min/max possible softmax loss \( L_i \)?

Q2: At initialization all \( s_j \) will be approximately equal; what is the softmax loss \( L_i \), assuming \( C \) classes?

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Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities

\[ s = f(x_i; W) \quad P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \]

Maximize probability of correct class

\[ L_i = -\log P(Y = y_i | X = x_i) \]

Putting it all together:

\[ L_i = -\log \left( \frac{e^{s_{y_i}}}{\sum_j e^{s_j}} \right) \]

Q: What is the min/max possible loss \( L_i \)?

A: min 0, max infinity
Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

\[ s = f(x_i; W) \]

\[ P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \]

Maximize probability of correct class

Putting it all together:

\[ L_i = -\log P(Y = y_i | X = x_i) \]

\[ L_i = -\log \left( \frac{e^{s_{y_i}}}{\sum_j e^{s_j}} \right) \]

Q2: At initialization all \( s \) will be approximately equal; what is the loss?
Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as probabilities:

\[ s = f(x_i; W) \quad P(Y = k|X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \]

Softmax Function

Maximize probability of correct class

Putting it all together:

\[ L_i = -\log P(Y = y_i|X = x_i) \quad L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right) \]

Q2: At initialization all s will be approximately equal; what is the loss?

A: \(-\log(1/C) = \log(C)\),

If \(C = 10\), then \(L_i = \log(10) \approx 2.3\)
Softmax vs. SVM

Softmax:
\[
\begin{align*}
\exp(Wx_i + b) &= \exp(-2.85 + 0.86) + \exp(-2.85 + 0.28) + \exp(-2.85 - 0.3) + \exp(-2.85 - 0.86 + 1) + \exp(-2.85 - 0.86 - 0.28 + 1) \\
&= 2.36 + 0.058 + 0.28 + 0.016 + 0.631 \\
&= 3.507
\end{align*}
\]

Normalize:
\[
\begin{align*}
\frac{\exp(Wx_i + b)}{\sum_{j=1}^{n} \exp(Wx_i + b_j)} &= \frac{2.36}{3.507} \\
&= 0.672
\end{align*}
\]

Hinge loss (SVM):
\[
\begin{align*}
\max(0, -2.85 - 0.28 + 1) + \\
\max(0, 0.86 - 0.28 + 1) &= 1.58
\end{align*}
\]
Softmax vs. SVM

\[ L_i = - \log \left( \frac{e^{s_{y_i}}}{\sum_j e^{s_j}} \right) \]  
\[ L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \]
Softmax vs. SVM

\[ L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right) \quad L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \]

assume scores:

[10, -2, 3]
[10, 9, 9]
[10, -100, -100]

and \( y_i = 0 \)

Q: What is the softmax loss and the SVM loss?
**Softmax vs. SVM**

\[ L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right) \quad \text{and} \quad L_i = \sum_{j\neq y_i} \max(0, s_j - s_{y_i} + 1) \]

Assume scores:

- [10, -2, 3]
- [10, 9, 9]
- [10, -100, -100]

and \( y_i = 0 \)

Q: What is the **softmax loss** and the **SVM loss** if I double the correct class score from 10 -> 20?
Recap

- We have some dataset of \((x,y)\)
- We have a **score function**: \(s = f(x; W) = Wx\)
- We have a **loss function**:

\[
L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)
\]

**Softmax**

\[
L_i = \sum_{j\neq y_i} \max(0, s_j - s_{y_i} + 1)
\]

**SVM**

\[
L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)
\]

**Full loss**
Recap

- We have some dataset of (x,y)
- We have a **score function**: \( s = f(x; W) = Wx \)
- We have a **loss function**:

\[
L_i = -\log\left( \frac{e^{s_{y_i}}}{\sum_j e^{s_j}} \right)
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**Softmax**

\[
L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)
\]

**SVM**

\[
L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)
\]

**Full loss**

How do we find the best W?
Optimization
Strategy #1: A first very bad idea solution: **Random search**

```python
# assume X_train is the data where each column is an example (e.g. 3073 x 50,000)
# assume Y_train are the labels (e.g. 1D array of 50,000)
# assume the function L evaluates the loss function

bestloss = float("inf")  # Python assigns the highest possible float value
for num in xrange(1000):
    W = np.random.randn(10, 3073) * 0.0001  # generate random parameters
    loss = L(X_train, Y_train, W)  # get the loss over the entire training set
    if loss < bestloss:  # keep track of the best solution
        bestloss = loss
        bestW = W
    print 'in attempt %d the loss was %f, best %f' % (num, loss, bestloss)

# prints:
# in attempt 0 the loss was 9.401632, best 9.401632
# in attempt 1 the loss was 8.959668, best 8.959668
# in attempt 2 the loss was 9.044034, best 8.959668
# in attempt 3 the loss was 9.278940, best 8.959668
# in attempt 4 the loss was 8.857370, best 8.857370
# in attempt 5 the loss was 8.943151, best 8.857370
# in attempt 6 the loss was 8.605604, best 8.605604
# ... (truncated: continues for 1000 lines)
```
Let's see how well this works on the test set...

```python
# Assume X_test is [3073 x 10000], Y_test [10000 x 1]
scores = Wbest.dot(Xte_cols) # 10 x 10000, the class scores for all test examples
# find the index with max score in each column (the predicted class)
Yte_predict = np.argmax(scores, axis = 0)
# and calculate accuracy (fraction of predictions that are correct)
np.mean(Yte_predict == Yte)
# returns 0.1555
```

15.5% accuracy! not bad! (SOTA is ~99.3%)
Strategy #2: Follow the slope
Strategy #2: Follow the slope

In 1-dimension, the derivative of a function:

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In multiple dimensions, the gradient is the vector of (partial derivatives) along each dimension.

The slope in any direction is the dot product of the direction with the gradient. The direction of steepest descent is the negative gradient.
<table>
<thead>
<tr>
<th>current $W$:</th>
<th>gradient $dW$:</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0.34, -1.11, 0.78, 0.12, 0.55, 2.81, -3.1, -1.5, 0.33, …]</td>
<td>[?, ?, ?, ?, ?, ?, ?, ?], […]</td>
</tr>
<tr>
<td>loss 1.25347</td>
<td></td>
</tr>
<tr>
<td>current $W$:</td>
<td>$W + h$ (first dim):</td>
</tr>
<tr>
<td>------------</td>
<td>----------------------</td>
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<td>[0.34, -1.11, 0.78, 0.12, 0.55, 2.81, -3.1, -1.5, 0.33, ...]</td>
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</tr>
<tr>
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<td>loss 1.25322</td>
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</table>
current W: [0.34, -1.11, 0.78, 0.12, 0.55, 2.81, -3.1, -1.5, 0.33, ...]  
loss 1.25347

W + h (first dim): [0.34 + 0.0001, -1.11, 0.78, 0.12, 0.55, 2.81, -3.1, -1.5, 0.33, ...]  
loss 1.25322

gradient dW: 

\[
\begin{align*}
\text{gradient } dW &= \frac{(1.25322 - 1.25347)/0.0001}{-2.5} \\
\end{align*}
\]

\[
\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
? , ? , ? , ...
\]
<table>
<thead>
<tr>
<th>current $W$:</th>
<th>$W + h$ (second dim):</th>
<th>gradient $dW$:</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0.34,</td>
<td>[0.34,</td>
<td>[-2.5,</td>
</tr>
<tr>
<td>-1.11,</td>
<td>-1.11 + 0.0001,</td>
<td>?,</td>
</tr>
<tr>
<td>0.78,</td>
<td>0.78,</td>
<td>?,</td>
</tr>
<tr>
<td>0.12,</td>
<td>0.12,</td>
<td>?,</td>
</tr>
<tr>
<td>0.55,</td>
<td>0.55,</td>
<td>?,</td>
</tr>
<tr>
<td>2.81,</td>
<td>2.81,</td>
<td>?,</td>
</tr>
<tr>
<td>-3.1,</td>
<td>-3.1,</td>
<td>?,</td>
</tr>
<tr>
<td>-1.5,</td>
<td>-1.5,</td>
<td>?,</td>
</tr>
<tr>
<td>0.33,...]</td>
<td>0.33,...]</td>
<td>?,</td>
</tr>
<tr>
<td>loss 1.25347</td>
<td>loss 1.25353</td>
<td>loss 1.25353</td>
</tr>
</tbody>
</table>
### current W:

| 0.34,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,... | 0.34,  
-1.11 + 0.0001,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,... |

**loss 1.25347**

### W + h (second dim):

| 0.34,  
-1.11 + 0.0001,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,... | 0.34,  
-1.11 + 0.0001,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,... |

**loss 1.25353**

### gradient dW:

[-2.5,  
0.6,  
?,  
?,  
?,  
?,  
?,  
?,...]

$$(1.25353 - 1.25347)/0.0001 = 0.6$$

\[
\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]
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<th>( W + h ) (third dim):</th>
<th>gradient ( dW ):</th>
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<td>([-2.5, 0.6, ?, ?, ?, ?, ?, ?, ?, \ldots]</td>
</tr>
<tr>
<td><strong>loss 1.25347</strong></td>
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### current W:

| 0.34,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,... |

**loss 1.25347**

### W + h (third dim):

| 0.34,  
-1.11,  
0.78 + **0.0001**,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,... |

**loss 1.25347**

### gradient dW:

| [-2.5,  
0.6,  
0,  
?,  
?,  
?,  
?,  
?,  
?,  
?,...] |

### Explanation:

The derivative of a function $f(x)$ is given by:

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
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<td>[-2.5, 0.6, 0, ?, ?, 0, ?, ..., ...]</td>
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**Numeric Gradient**
- Slow! Need to loop over all dimensions
- Approximate

loss 1.25347

loss 1.25347
This is silly. The loss is just a function of $W$:

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + \sum_k W_k^2$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$s = f(x; W) = Wx$$

want $\nabla_W L$
This is silly. The loss is just a function of $W$:

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + \sum_{k} W_k^2$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$s = f(x; W) = Wx$$

want $\nabla_W L$

Use calculus to compute an analytic gradient
current $W$: 

$[0.34, 
-1.11, 
0.78, 
0.12, 
0.55, 
2.81, 
-3.1, 
-1.5, 
-1.5, 
0.33, ...]$ 

loss $1.25347$

gradient $dW$: 

$[-2.5, 
0.6, 
0, 
0.2, 
0.7, 
-0.5, 
1.1, 
1.3, 
-2.1, ...]$ 

d$W = ...$

(some function data and $W$)
In summary:

- Numerical gradient: approximate, slow, easy to write
- Analytic gradient: exact, fast, error-prone

=>

In practice: Always use analytic gradient, but check implementation with numerical gradient. This is called a gradient check.
Gradient Descent

# Vanilla Gradient Descent

```python
while True:
    weights_grad = evaluate_gradient(loss_fun, data, weights)
    weights += - step_size * weights_grad # perform parameter update
```
negative gradient direction

original $W$

$W_1$

$W_2$
Stochastic Gradient Descent (SGD)

\[ L(W) = \frac{1}{N} \sum_{i=1}^{N} L_i(x_i, y_i, W) + \lambda R(W) \]

\[ \nabla_W L(W) = \frac{1}{N} \sum_{i=1}^{N} \nabla_W L_i(x_i, y_i, W) + \lambda \nabla_W R(W) \]

Full sum expensive when N is large!

Approximate sum using a minibatch of examples
32 / 64 / 128 common

# Vanilla Minibatch Gradient Descent

```python
while True:
    data_batch = sample_training_data(data, 256) # sample 256 examples
    weights_grad = evaluate_gradient(loss_fun, data_batch, weights)
    weights += - step_size * weights_grad # perform parameter update
```
Interactive Web Demo

http://vision.stanford.edu/teaching/cs231n-demos/linear-classify/
Next time:

Introduction to neural networks

Backpropagation