Review:
Linear Algebra for Camera Models

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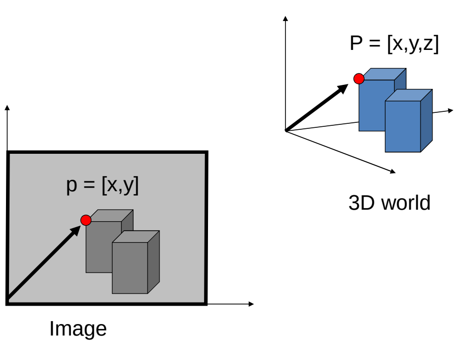
Stanford Vision Lab

Agenda

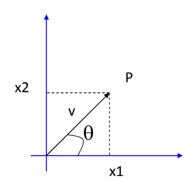
- Basic definitions and properties
- Geometrical transformations

Reading:
[HZ] Chapter 2, 4

Vectors (i.e., 2D or 3D vectors)



Vectors (i.e., 2D vectors)



$$\mathbf{v} = (x_1, x_2)$$

Magnitude: $\| \mathbf{v} \| = \sqrt{x_1^{\ 2} + x_2^{\ 2}}$

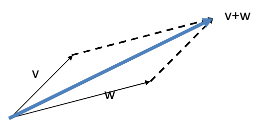
If $\| \mathbf{v} \| = 1$, $\mathbf{v}$ Is a UNIT vector

$\dfrac{\mathbf{v}}{\| \mathbf{v} \|} = \left(\dfrac{x_1}{\| \mathbf{v} \|}, \dfrac{x_2}{\| \mathbf{v} \|} \right)$ Is a unit vector

Orientation: $\theta = \tan^{-1}\left(\dfrac{x_2}{x_1} \right)$

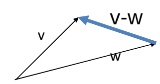
Vector Addition

$$\mathbf{v}+\mathbf{w}=(x_1,x_2)+(y_1,y_2)=(x_1+y_1,x_2+y_2)$$



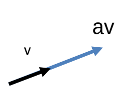
Vector Subtraction

$$\mathbf{v}-\mathbf{w}=(x_1,x_2)-(y_1,y_2)=(x_1-y_1,x_2-y_2)$$

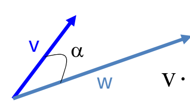


Scalar Product

$$a\mathbf{v}=a(x_1,x_2)=(ax_1,ax_2)$$



Inner (dot) Product



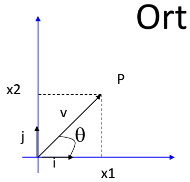
$$\mathrm{v}\cdot\mathrm{w}=(\mathrm{x}_1,\mathrm{x}_2)\cdot(\mathrm{y}_1,\mathrm{y}_2)=\mathrm{x}_1\mathrm{y}_1+\mathrm{x}_2\mathrm{y}_2$$

The inner product is a SCALAR!

$$\mathrm{v}\cdot\mathrm{w}=(\mathrm{x}_1,\mathrm{x}_2)\cdot(\mathrm{y}_1,\mathrm{y}_2)=\|\mathrm{v}\|\cdot\|\mathrm{w}\|\cos\boldsymbol{\alpha}$$

$$\text{if}\quad \mathrm{v}\perp\mathrm{w},\quad \mathrm{v}\cdot\mathrm{w}=?\ =0$$

Orthonormal Basis

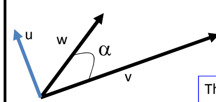


$$\mathbf{i}=(1,0) \qquad \|\mathbf{i}\|=1 \qquad \mathbf{i}\cdot\mathbf{j}=0$$
$$\mathbf{j}=(0,1) \qquad \|\mathbf{j}\|=1$$

$$\mathbf{v}=(x_1,x_2) \qquad \mathbf{v}=\mathrm{x}_1\mathbf{i}+\mathrm{x}_2\mathbf{j}$$

$$\mathbf{v}\cdot\mathbf{i}=?\;=(\mathrm{x}_1\mathbf{i}+\mathrm{x}_2\mathbf{j})\cdot\mathbf{i}=\mathrm{x}_1 1+\mathrm{x}_2 0=\mathrm{x}_1$$
$$\mathbf{v}\cdot\mathbf{j}=(\mathrm{x}_1\mathbf{i}+\mathrm{x}_2\mathbf{j})\cdot\mathbf{j}=\mathrm{x}_1.0+\mathrm{x}_2.1=\mathrm{x}_2$$

Vector (cross) Product



$$u=v\times w$$

The cross product is a VECTOR!

Magnitude: $\|\mathrm{u}\|=\|\mathrm{v}\cdot\mathrm{w}\|=\|\mathrm{v}\|\|\mathrm{w}\|\sin\alpha$

Orientation:
$$u\perp v\Rightarrow u\cdot v=(v\times w)\cdot v=0$$
$$u\perp w\Rightarrow u\cdot w=(v\times w)\cdot w=0$$

$$\text{if} \quad \mathrm{v}/\!/\mathrm{w}\;? \quad \rightarrow \mathrm{u}=0$$

Vector Product Computation

$\mathbf{i}=(1,0,0) \qquad \|\mathbf{i}\|=1$

$\mathbf{j}=(0,1,0) \qquad \|\mathbf{j}\|=1 \qquad \mathbf{i}\cdot\mathbf{j}=0 \quad \mathbf{i}\cdot\mathbf{k}=0 \quad \mathbf{j}\cdot\mathbf{k}=0$

$\mathbf{k}=(0,0,1) \qquad \|\mathbf{k}\|=1$

$$\mathbf{u}=\mathbf{v}\times\mathbf{w}=(x_1,x_2,x_3)\times(y_1,y_2,y_3)$$
$$=(x_2y_3-x_3y_2)\mathbf{i}+(x_3y_1-x_1y_3)\mathbf{j}+(x_1y_2-x_2y_1)\mathbf{k}$$

Matrices

$$A_{n\times m}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \boxed{a_{nm}} \end{bmatrix} \longleftrightarrow$$

Pixel's intensity value

Sum: $C_{n\times m}=A_{n\times m}+B_{n\times m} \qquad c_{ij}=a_{ij}+b_{ij}$

A and B must have the same dimensions!

Example: $\begin{bmatrix}2&0\\0&1\end{bmatrix}+\begin{bmatrix}0&2\\1&0\end{bmatrix}=?\;=\begin{bmatrix}2&2\\1&1\end{bmatrix} \qquad =\begin{bmatrix}\mathbf{I}&\mathbf{t}\\0&1\end{bmatrix}\cdot\mathbf{P}=\mathbf{T}\cdot\mathbf{P}$

Matrices

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S & \mathbf{t} \\ 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$t = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Matrices

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \quad \mathbf{a}_{\mathrm{i}}$$

$$B_{n\times m} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ b_{31} & b_{32} & \cdots & b_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \quad \mathbf{b}_{\mathrm{j}}$$

Product:

$$C_{n\times p} = A_{n\times \boxed{m}} B_{\boxed{m}\times p}$$

$$\mathrm{c}_{\mathrm{ij}} = \mathbf{a}_{\mathrm{i}} \cdot \mathbf{b}_{\mathrm{j}} = \sum_{\mathrm{k}=1}^{\mathrm{m}} \mathrm{a}_{\mathrm{ik}} \mathrm{b}_{\mathrm{kj}}$$

A and B must have compatible dimensions!

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 3 & 0 \end{bmatrix}$$

Matrices

Transpose:

$$C_{m\times n} = A^T{}_{n\times m} \qquad (A+B)^T = A^T + B^T$$

$$c_{ij} = a_{ji} \qquad (AB)^T = B^T A^T$$

If $A^T = A$ A is symmetric

Examples:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} 6 & 2 \\ 1 & 5 \\ 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

$\begin{bmatrix} 5 & 2 \\ 1 & 5 \end{bmatrix}$ Symmetric? No!

$\begin{bmatrix} 3 & 2 \\ 2 & 7 \end{bmatrix}$ Symmetric? Yes!

Matrices

Determinant:

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

A must be square

Example: $\det\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13$

Matrices

Inverse: A must be square

$$A_{n\times n}A^{-1}_{n\times n} = A^{-1}_{n\times n}A_{n\times n} = I$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = ? \quad = \frac{1}{28}\begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28}\begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28}\begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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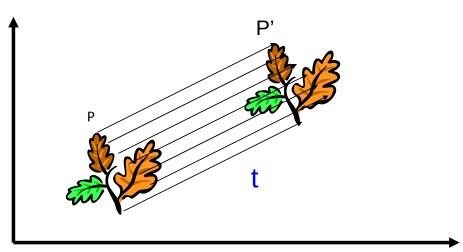
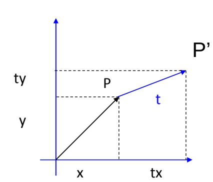
Reading:
[HZ] Chapter 2, 4

2D Translation



2D Translation Equation

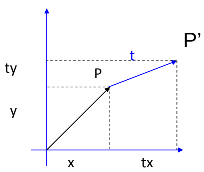


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = \mathbf{P} + \mathbf{t} = (x + t_x, y + t_y)$$

2D Translation using Matrices



$$\mathbf{P} = (x, y)$$
$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- Multiply the coordinates by a non-zero scalar and add an extra coordinate equal to that scalar. For example,

$$(x, y) \rightarrow (x \cdot z, y \cdot z, z) \quad z \neq 0$$
$$(x, y, z) \rightarrow (x \cdot w, y \cdot w, z \cdot w, w) \quad w \neq 0$$

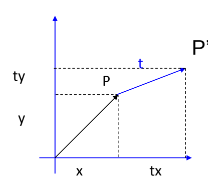
Back to Cartesian Coordinates:

- Divide by the last coordinate and eliminate it. For example,

$$(x, y, z) \quad z \neq 0 \rightarrow (x/z, y/z)$$
$$(x, y, z, w) \quad w \neq 0 \rightarrow (x/w, y/w, z/w)$$

- NOTE: in our example the scalar was 1

2D Translation using Homogeneous Coordinates



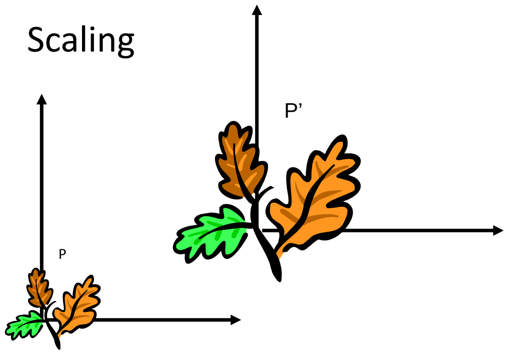
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$
$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

t

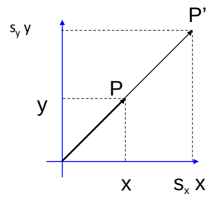
P

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}$$

Scaling



Scaling Equation



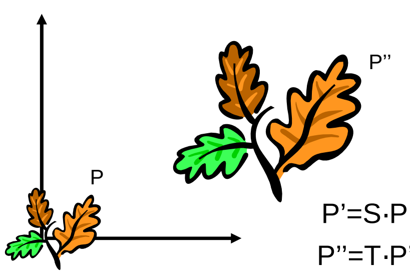
$$\mathbf{P} = (\mathrm{x}, \mathrm{y}) \rightarrow \mathbf{P'} = (\mathrm{s_x x}, \mathrm{s_y y})$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P'} = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P'} \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S'} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

Scaling & Translating



P'=S·P

P''=T·P'

P''=T · P'=T ·(S · P)= T · S ·P = A · P

Scaling & Translating

$$\mathbf{P''} = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\mathrm{A}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

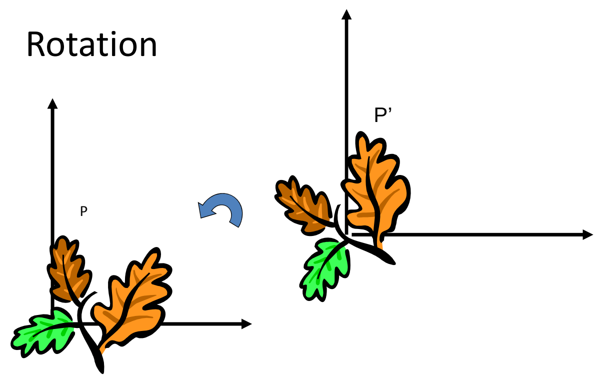
Translating & Scaling = Scaling & Translating ?

$$\mathbf{P'''} = \mathbf{T}\cdot\mathbf{S}\cdot\mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

$$\mathbf{P'''} = \mathbf{S}\cdot\mathbf{T}\cdot\mathbf{P} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

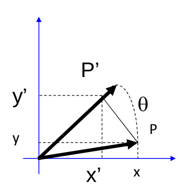
$$= \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

Rotation



Rotation Equations

Counter-clockwise rotation by an angle θ



$$x' = \cos\theta\, x - \sin\theta\, y$$
$$y' = \cos\theta\, y + \sin\theta\, x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R}\ \mathbf{P}$$

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2 → 4 elements

Note: R belongs to the category of *normal* matrices and satisfies many interesting properties:

$$\mathbf{R}\cdot\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\cdot\mathbf{R} = \mathbf{I}$$
$$\det(\mathbf{R}) = 1$$

Rotation+ Scaling +Translation

P'= (T R S) P

$$\mathbf{P'} = \mathbf{T}\cdot\mathrm{R}\cdot\mathbf{S}\cdot\mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} R' & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} R'S & t \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

If s_x=s_y, this is a similarity transformation!

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Transformation in 2D

-Isometries

-Similarities

-Affinity

-Projective

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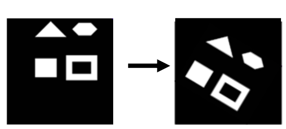
Transformation in 2D

Isometries:

[Euclideans]

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_e \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Preserve distance (areas)
- 3 DOF
- Regulate motion of rigid object



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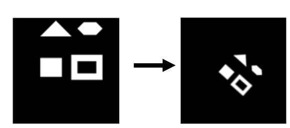
Transformation in 2D

Similarities:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\,R & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_s \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Preserve
  - ratio of lengths
  - angles
- -4 DOF



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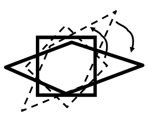
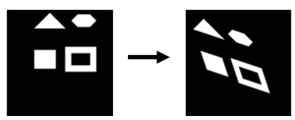
Transformation in 2D

Affinities:
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_a \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = R(\theta) \cdot R(-\phi) \cdot D \cdot R(\phi) \qquad D = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$





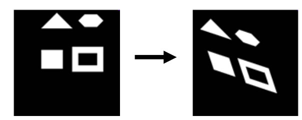
Transformation in 2D

Affinities:
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_a \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = R(\theta) \cdot R(-\phi) \cdot D \cdot R(\phi) \qquad D = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

-Preserve:
- Parallel lines
- Ratio of areas
- Ratio of lengths on collinear lines
- others…

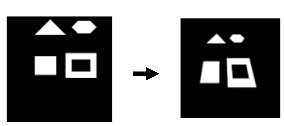
- 6 DOF



Transformation in 2D

Projective:
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ \boxed{v} & b \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_p \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- 8 DOF
- Preserve:
  - cross ratio of 4 collinear points
  - collinearity
  - and a few others…



Agenda

- Basic definitions and properties
- Geometrical transformations

Reading:
[HZ] Chapter 2, 4