Lecture 9: Epipolar Geometry

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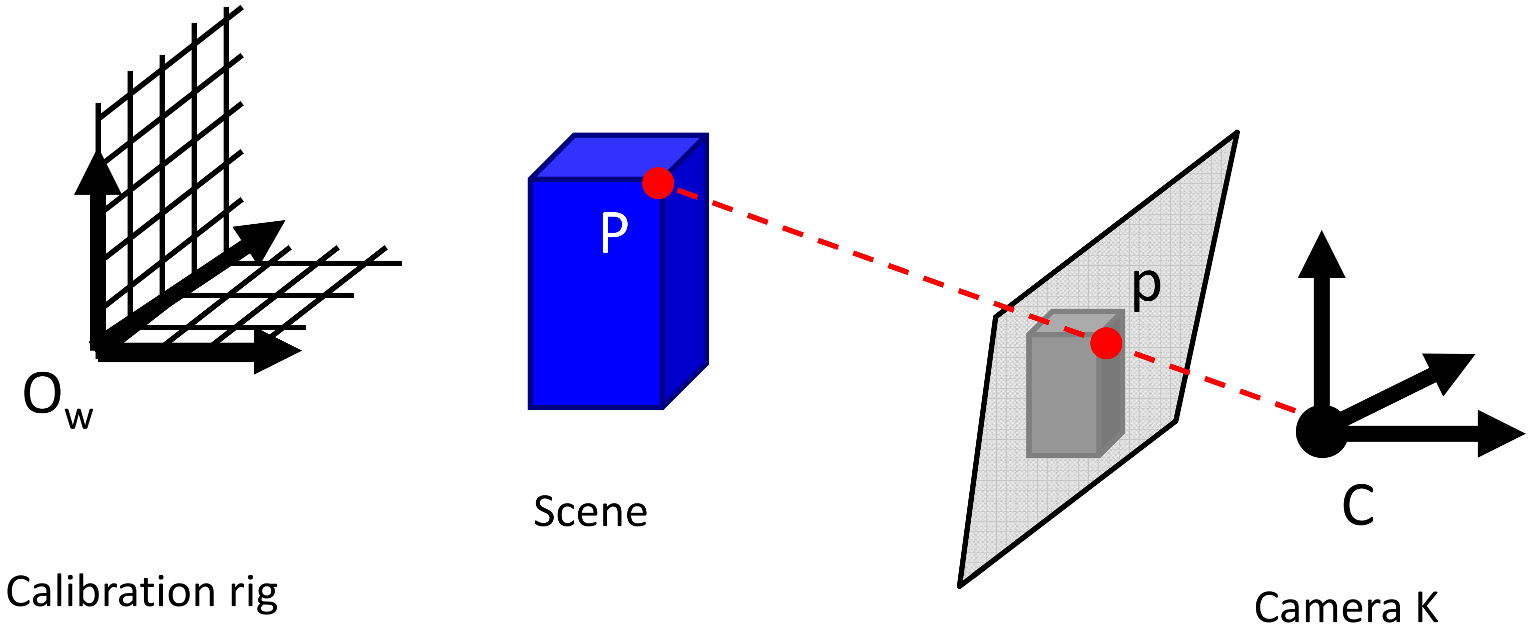
Stanford Vision Lab

What we will learn today?

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F (Problem Set 2 (Q2))
- Rectification

Reading:
[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10

Recovering structure from a single view



Why is it so difficult?

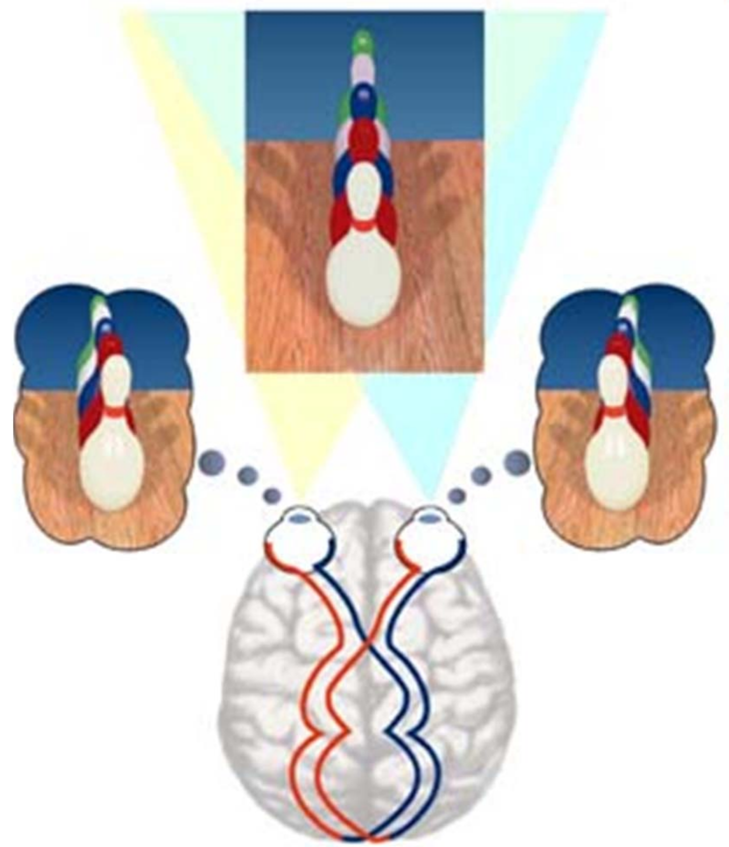
Intrinsic ambiguity of the mapping from 3D to image (2D)

Recovering structure from a single view

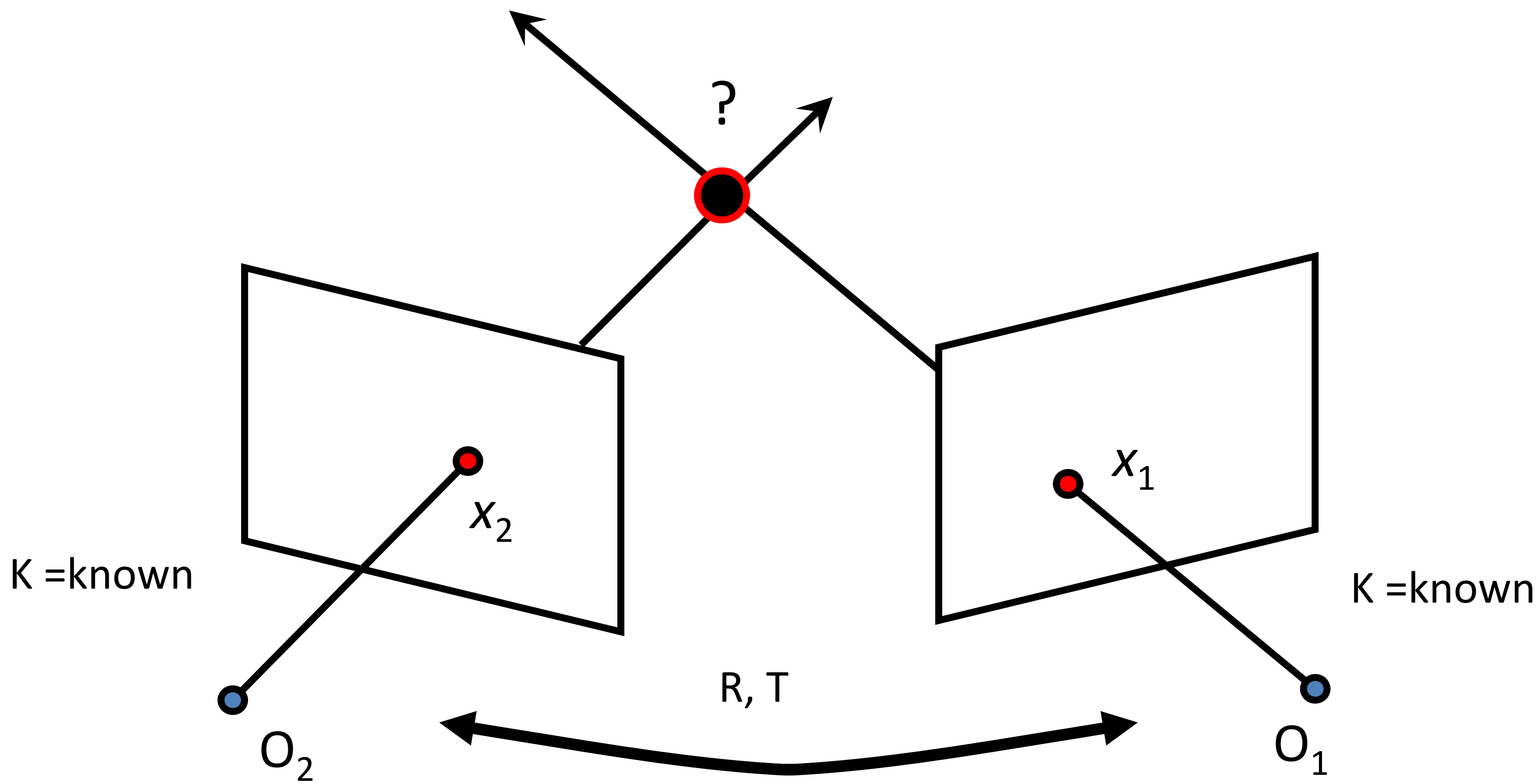
Courtesy slide S. Lazebnik

Intrinsic ambiguity of the mapping from 3D to image (2D)

Two eyes help!



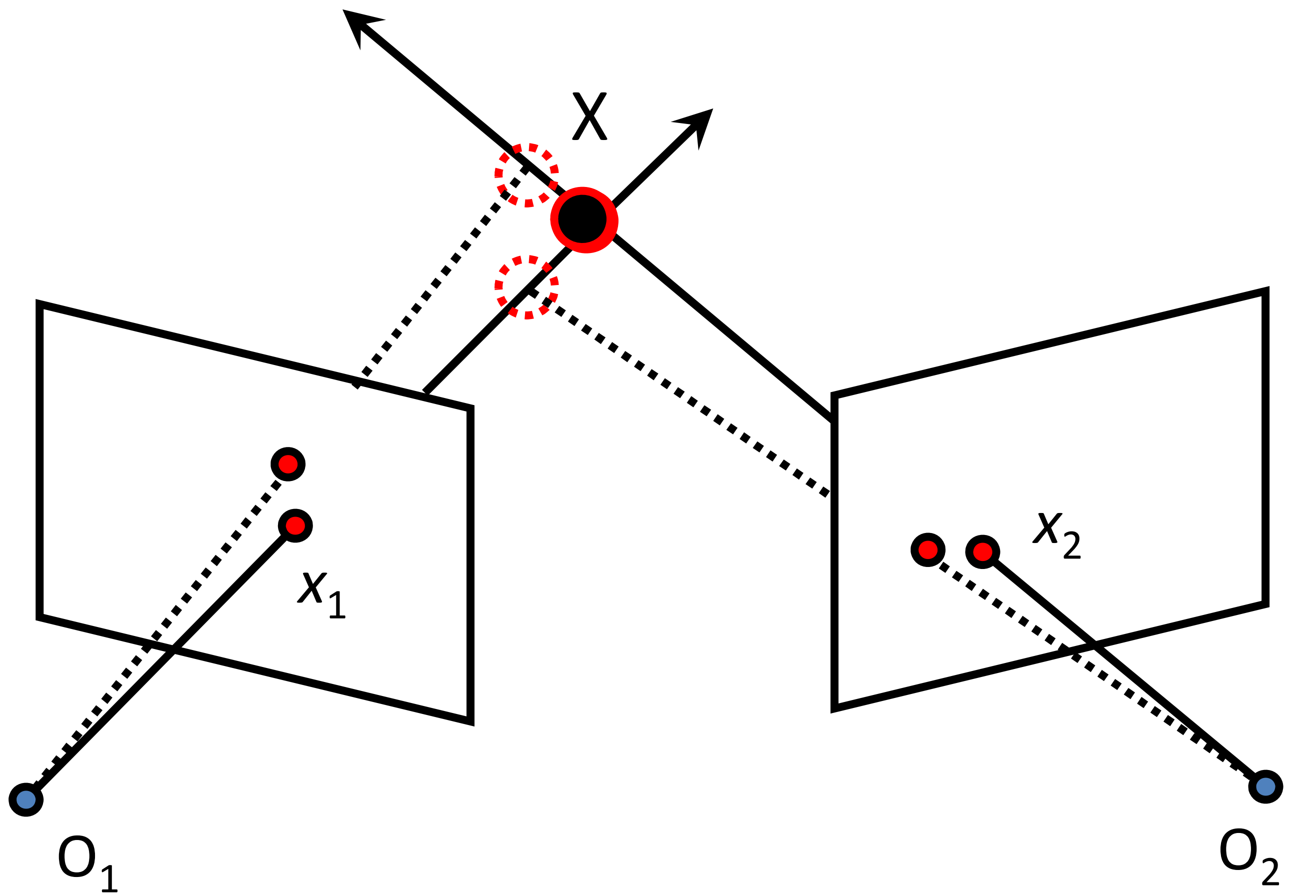
Two eyes help!



This is called triangulation

Triangulation

- Find X that minimizes $d^2(x_1, P_1X) + d^2(x_2, P_2X)$



Stereo-view geometry

- **Correspondence:** Given a point in one image, how can I find the corresponding point x' in another one?

- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.

- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.

This lecture (#9)

Stereo-view geometry

Next lecture (#10)

- **Correspondence:** Given a point in one image, how can I find the corresponding point x’ in another one?

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- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.

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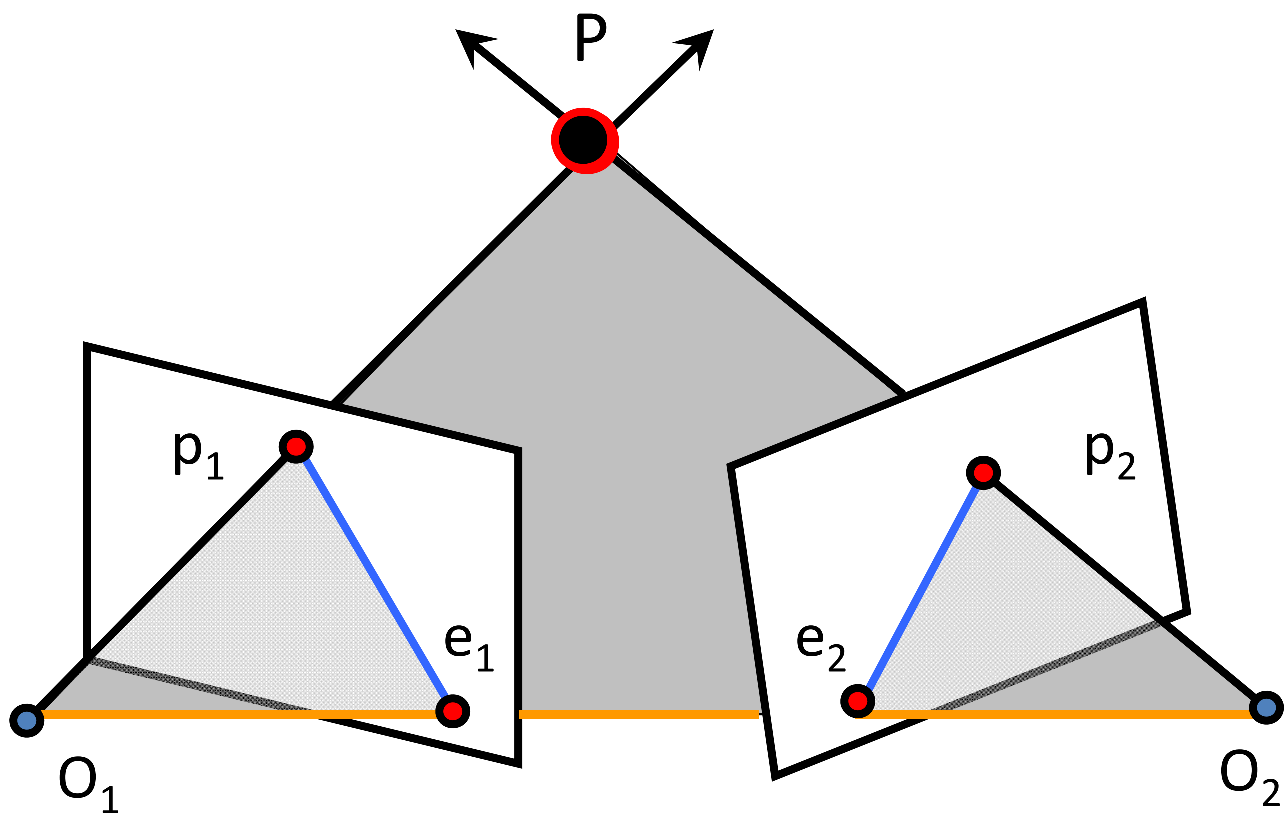
Epipolar geometry

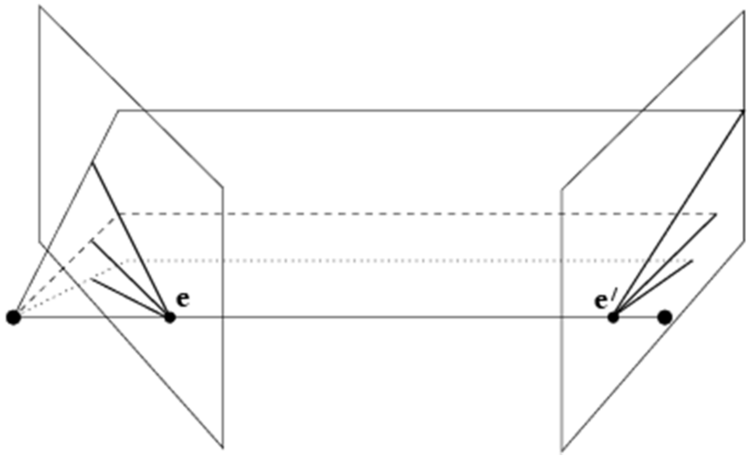


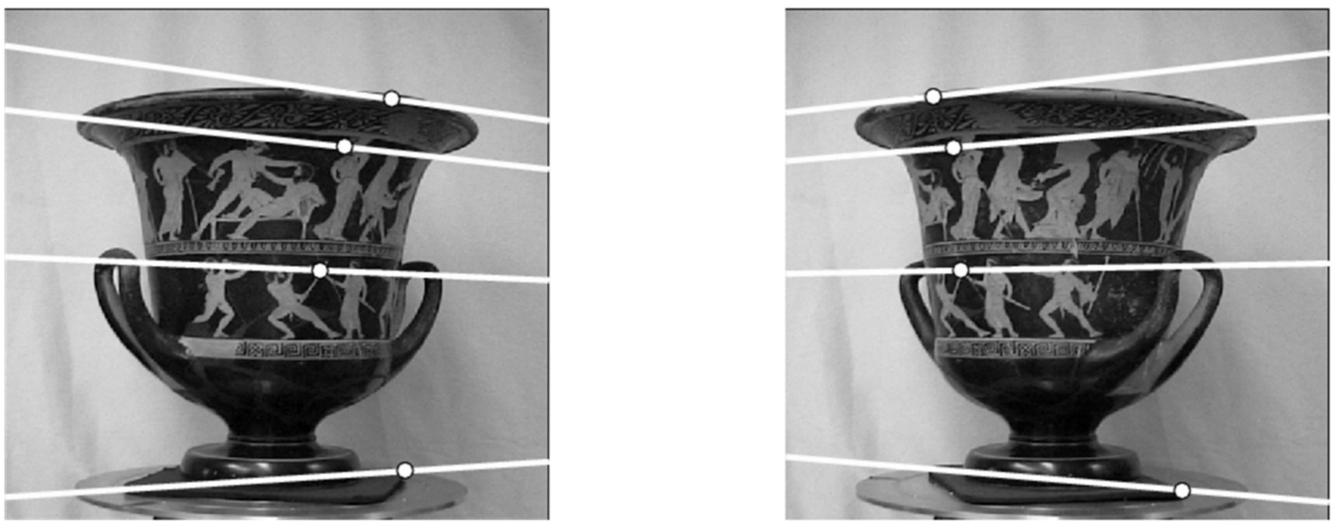
- Epipolar Plane
- Baseline
- Epipolar Lines

- Epipoles e_1, e_2

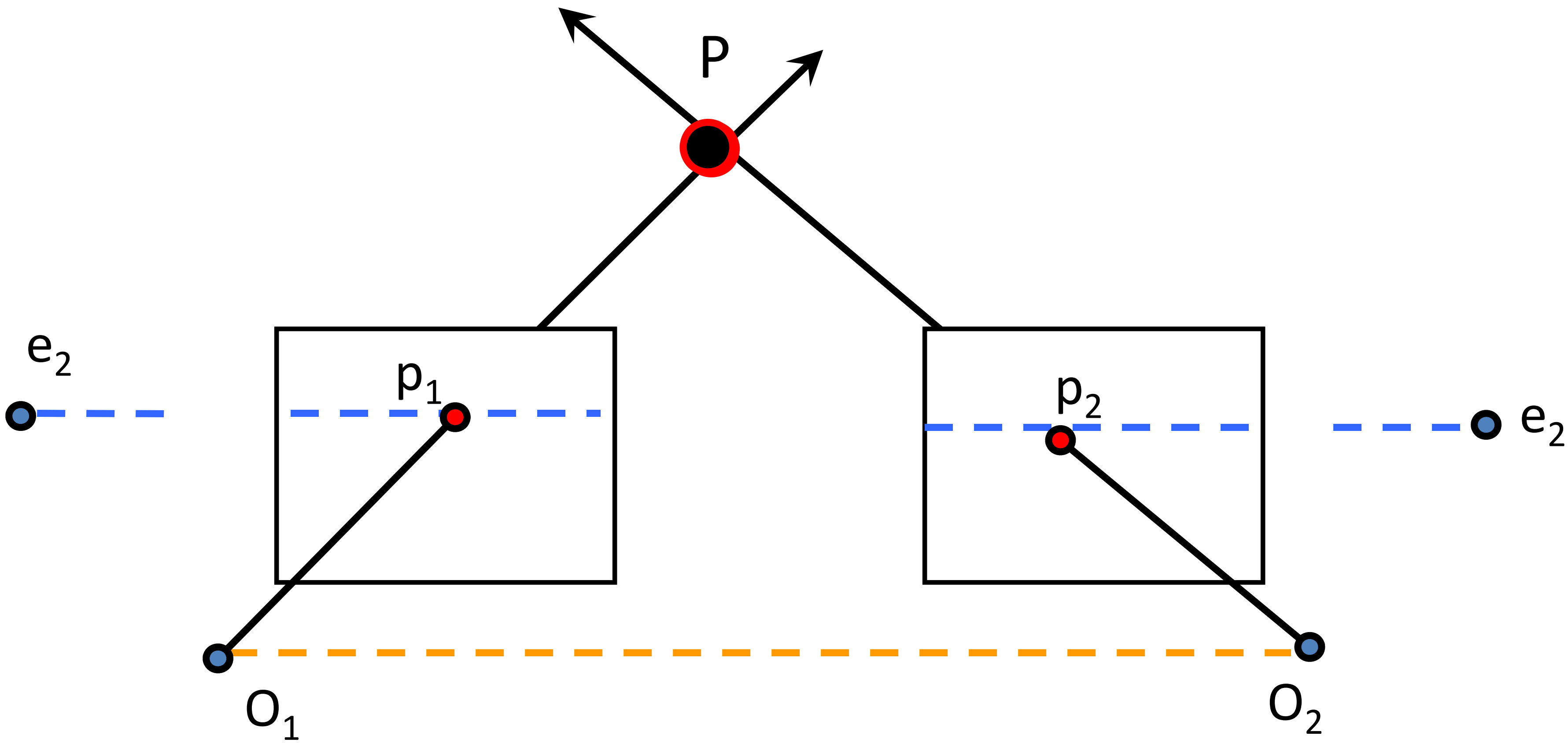
  = intersections of baseline with image planes
  = projections of the other camera center
  = vanishing points of camera motion direction

Example: Converging image planes



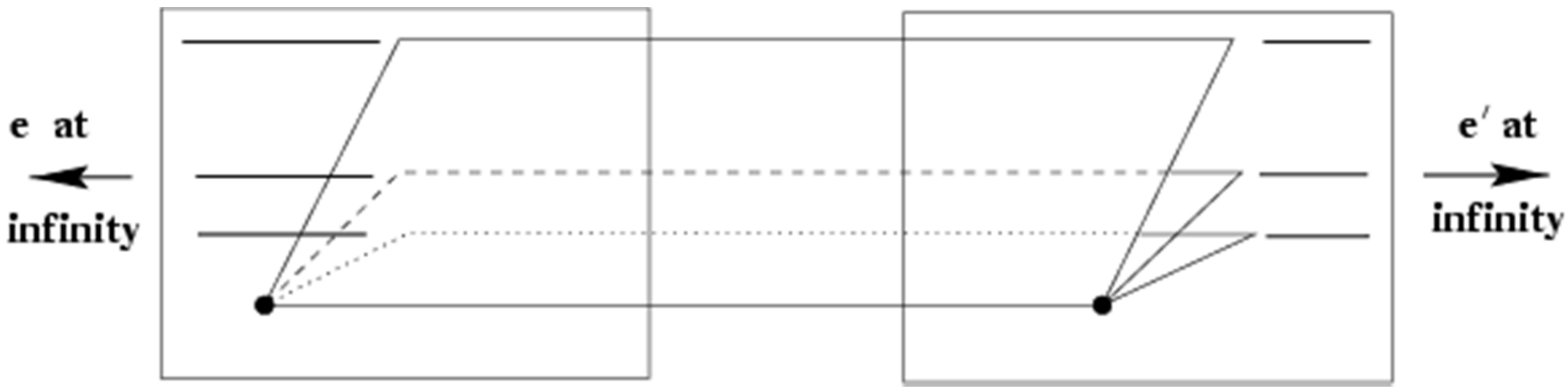


Example: Parallel image planes

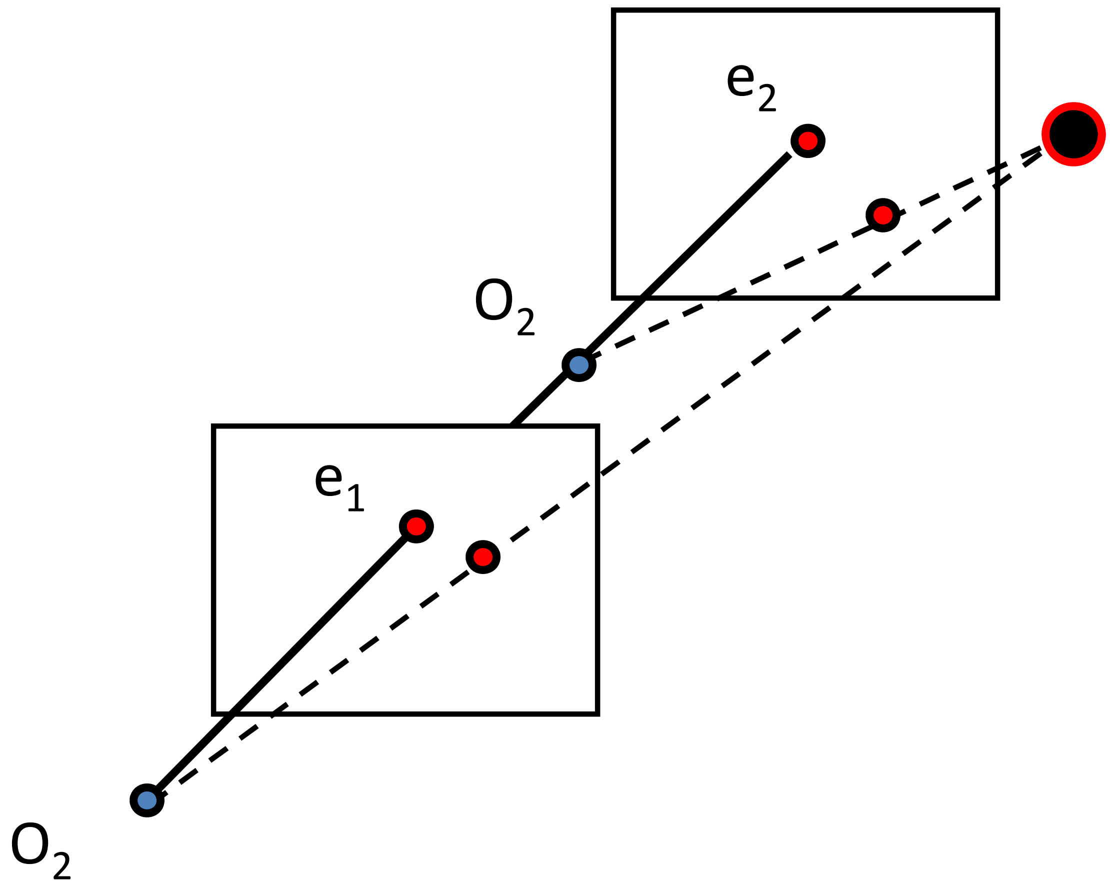


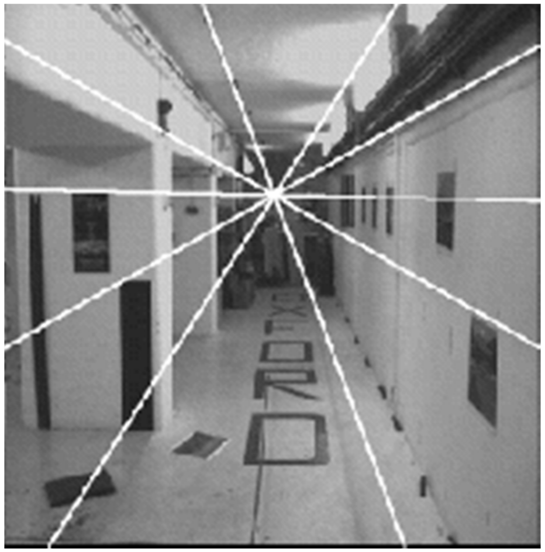
- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to x axis

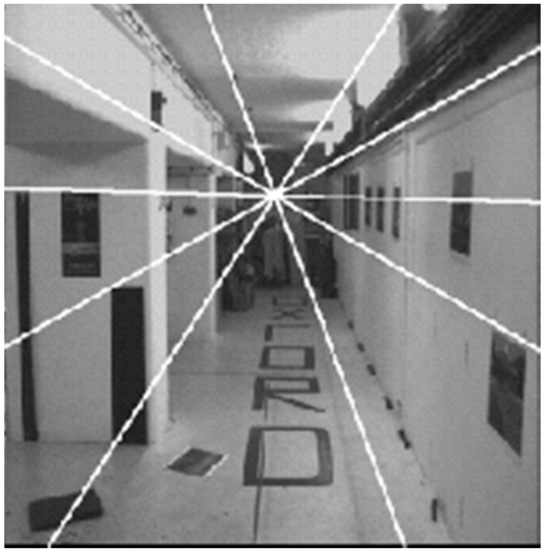
Example: Parallel image planes



Example: Forward translation

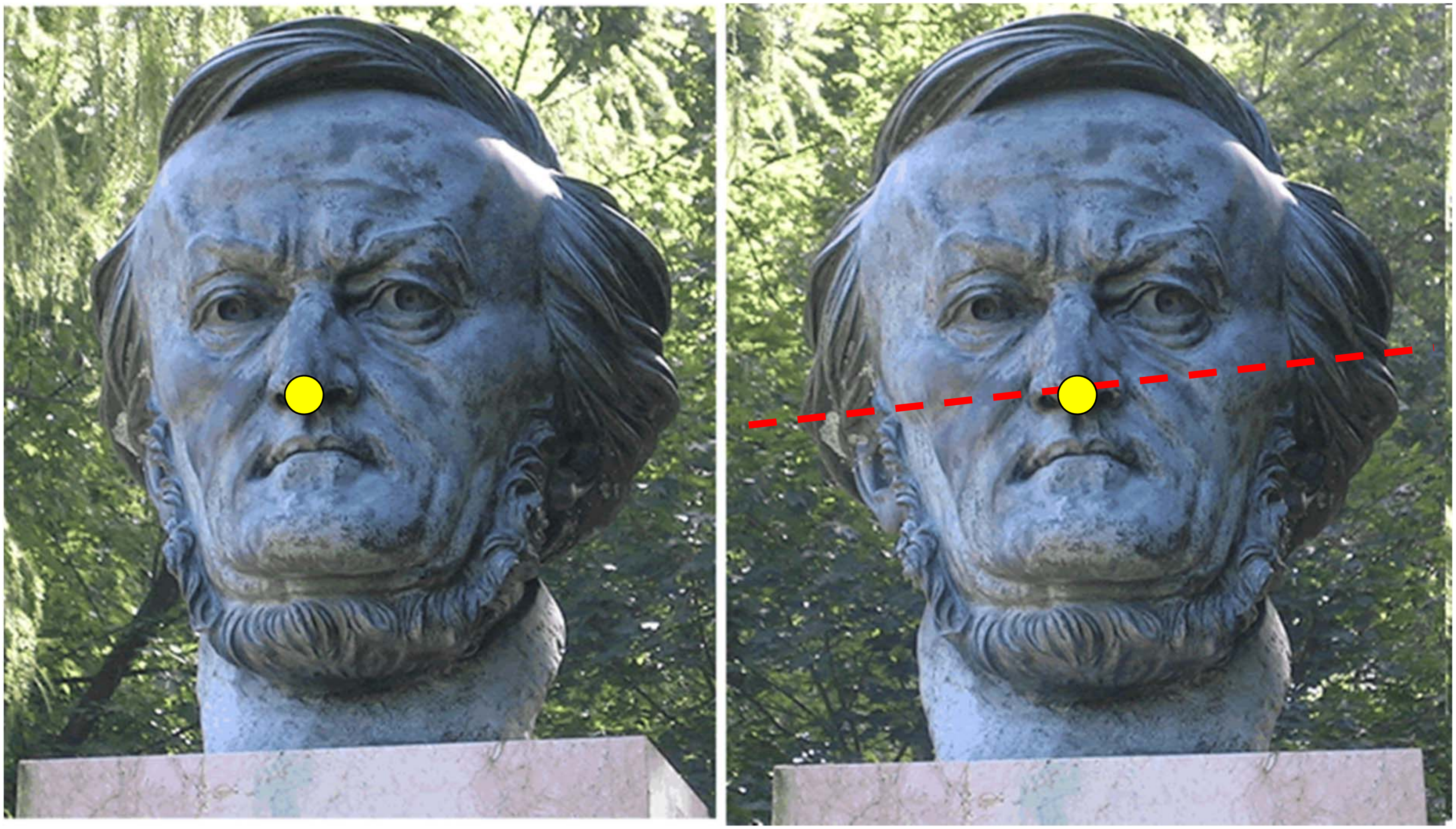






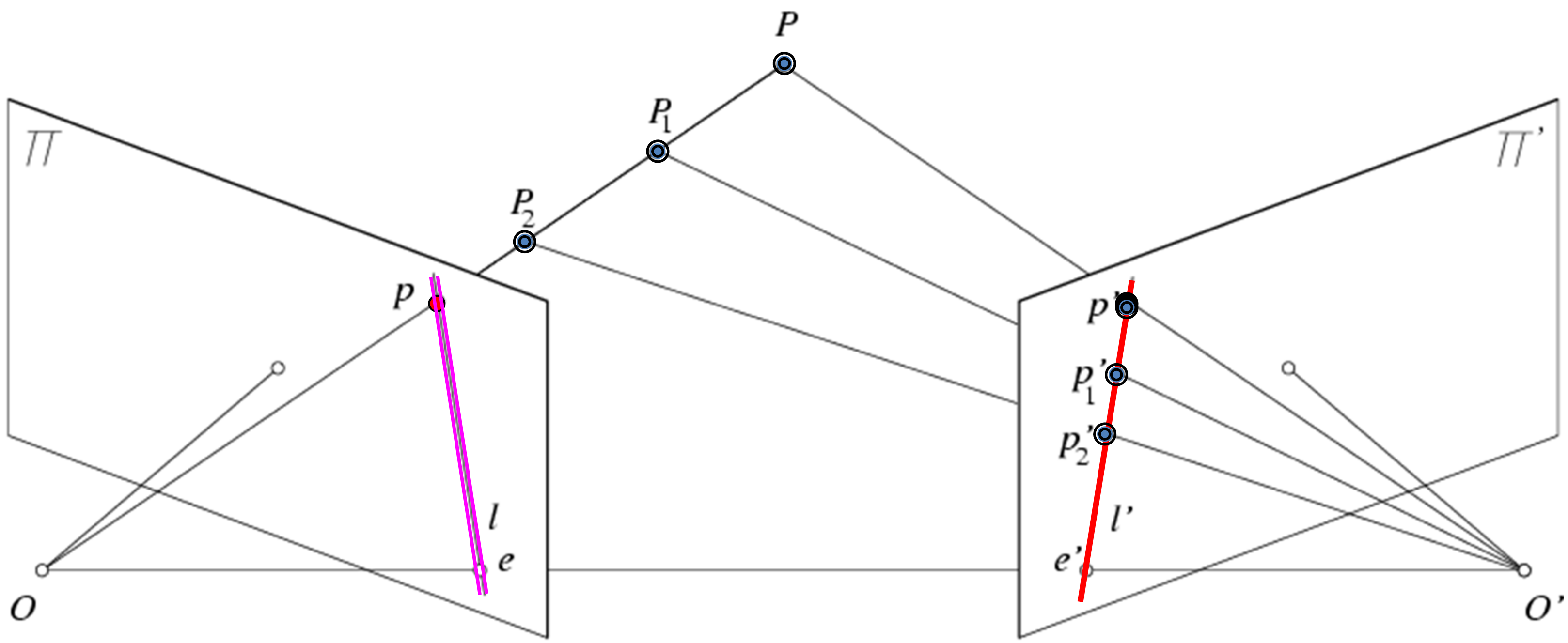
- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)

Epipolar Constraint



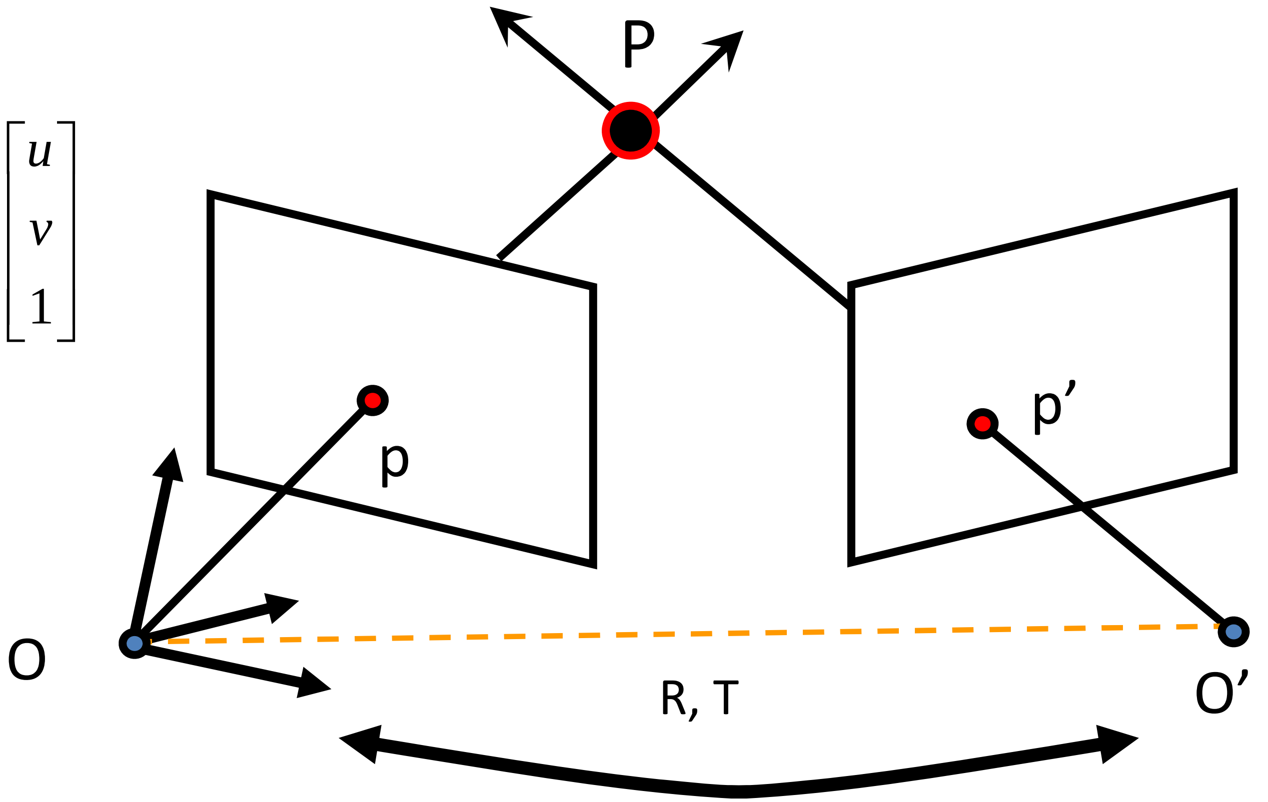
- Two views of the same object
- Suppose I know the camera positions and camera matrices
- Given a point on left image, how can I find the corresponding point on right image?

Epipolar Constraint



- Potential matches for *p* have to lie on the corresponding epipolar line *l'*.
- Potential matches for *p'* have to lie on the corresponding epipolar line *l*.

Epipolar Constraint



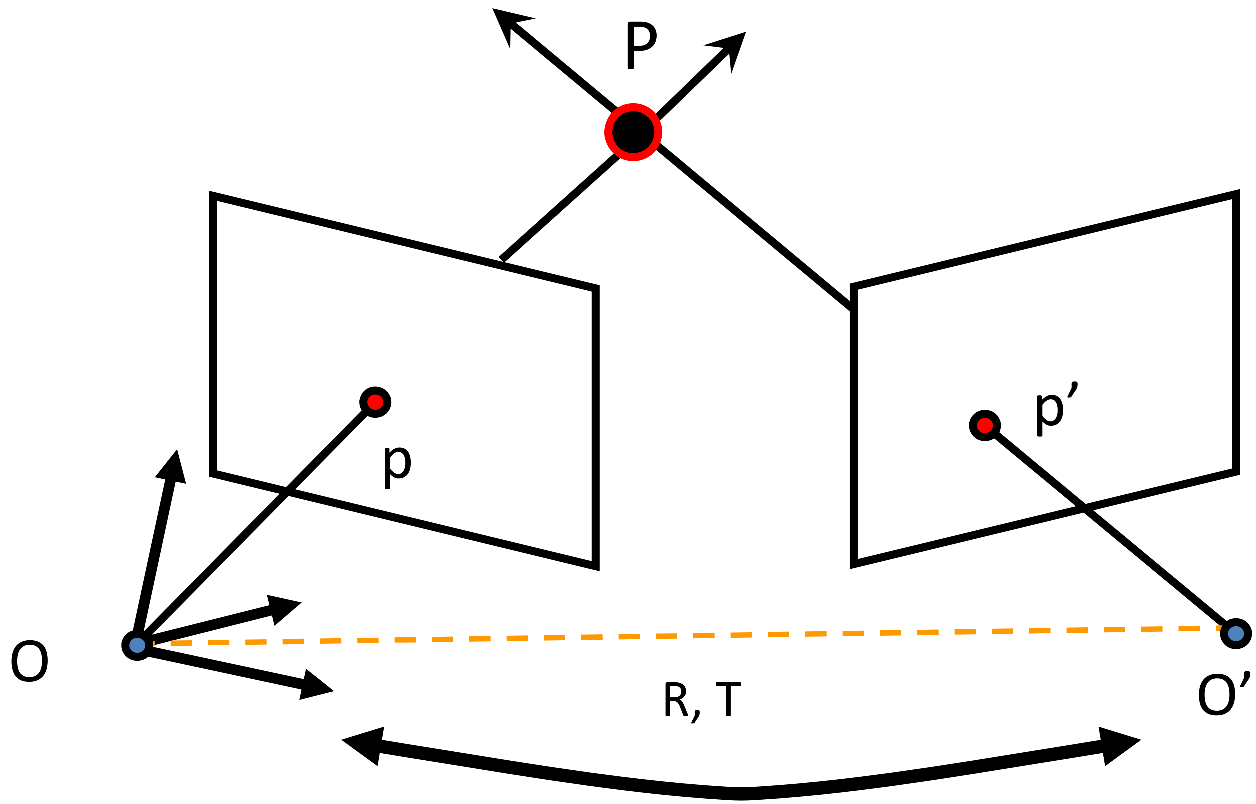
$$p \rightarrow M\ P = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$p \rightarrow M'\ P = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

$$\mathrm{M} = \mathrm{K}\begin{bmatrix} \mathrm{I} & 0 \end{bmatrix}$$

$$\mathrm{M}' = \mathrm{K}\begin{bmatrix} \mathrm{R} & \mathrm{T} \end{bmatrix}$$

Epipolar Constraint



$$M = K[I \quad 0]$$

$$\downarrow$$

$$M = [I \quad 0]$$

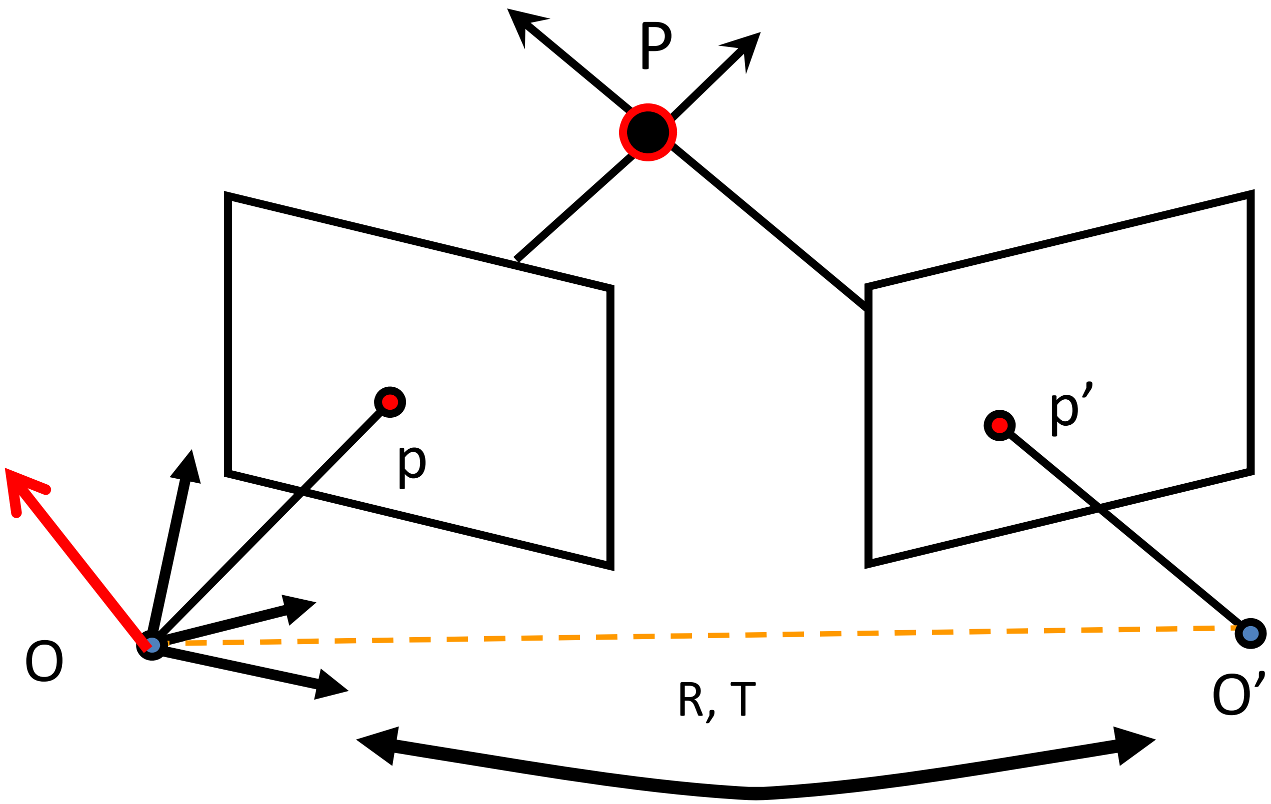
K_1 and K_2 are known (calibrated cameras)

$$M' = K[R \quad T]$$

$$\downarrow$$

$$M' = [R \quad T]$$

Epipolar Constraint



$$T \times (R\ p')$$

Perpendicular to epipolar plane

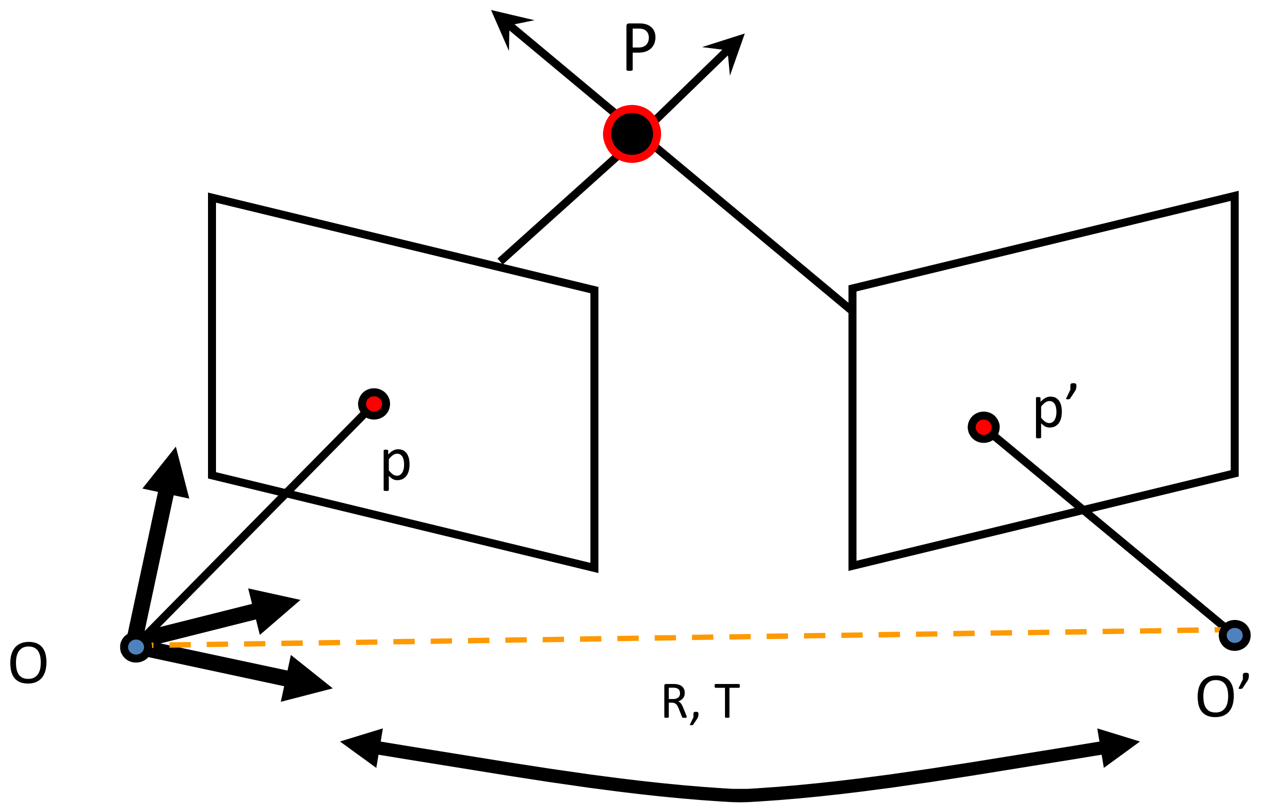
$$p^T \cdot [T \times (R\ p')] = 0$$

Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

"skew symmetric matrix"

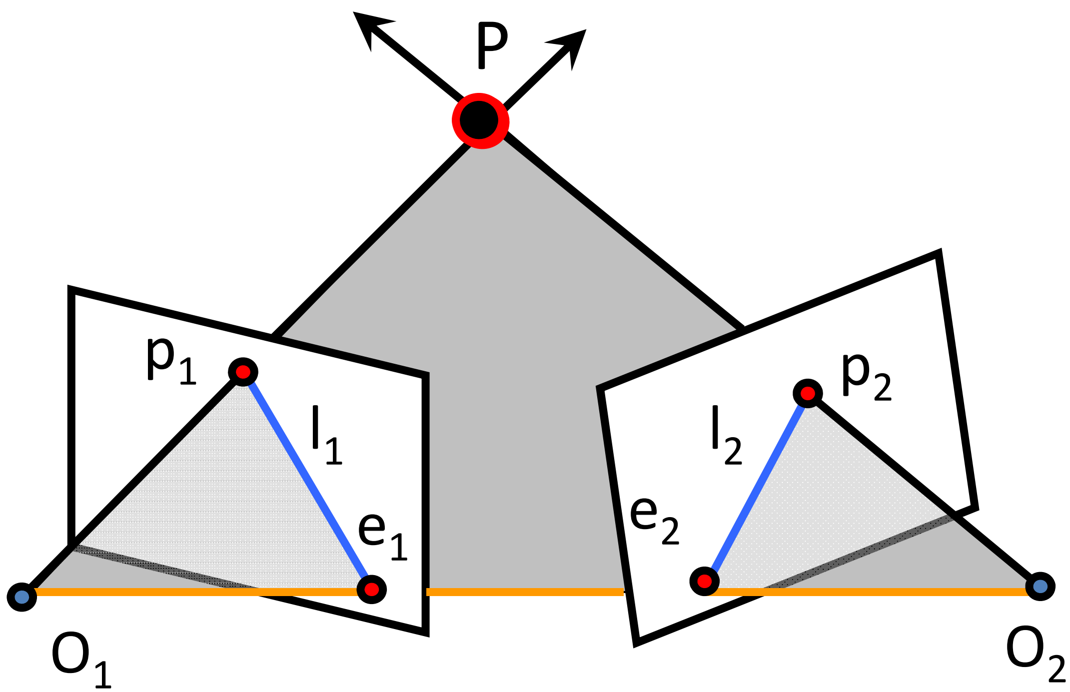
Triangulation



$$p^T \cdot [T \times (R\,p')] = 0 \rightarrow p^T \cdot \boxed{[T_\times] \cdot R}\, p' = 0$$

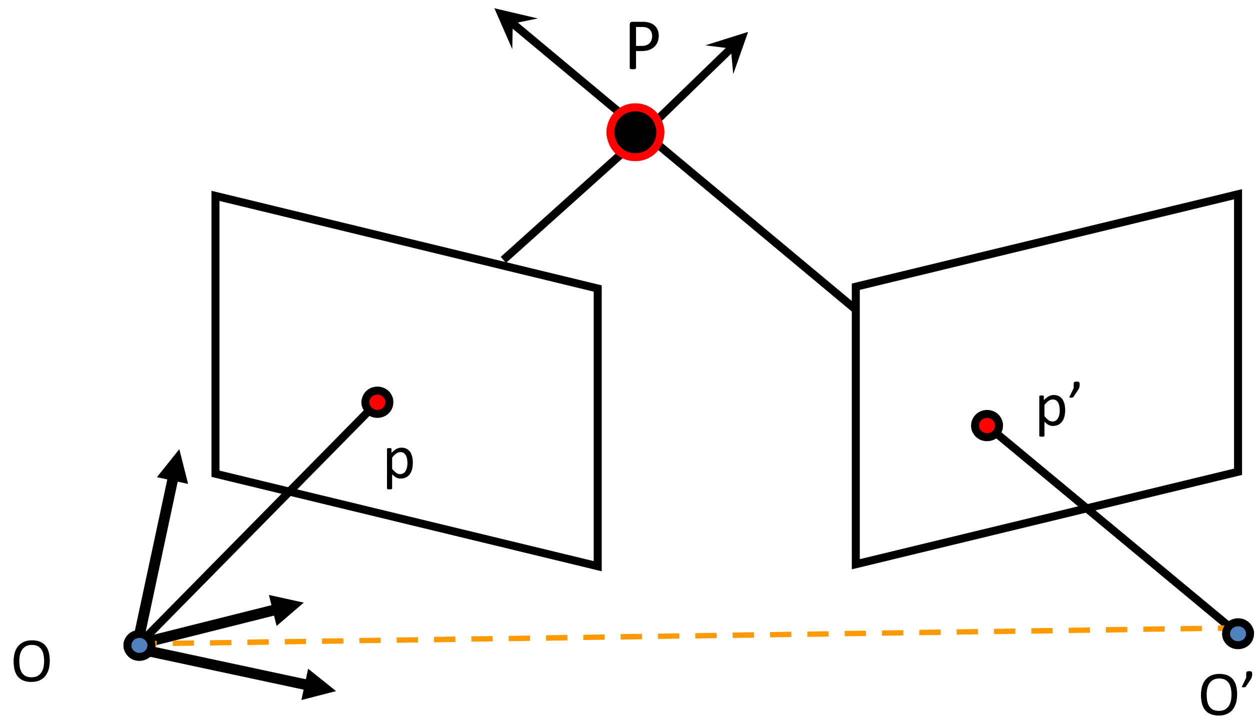
(Longuet-Higgins, 1981) E = essential matrix

Triangulation



- $E\, p_2$ is the epipolar line associated with p_2 ($l_1 = E\, p_2$)
- $E^T p_1$ is the epipolar line associated with p_1 ($l_2 = E^T p_1$)
- E is singular (rank two)
- $E\, e_2 = 0$ and $E^T e_1 = 0$
- E is 3x3 matrix; 5 DOF

Triangulation

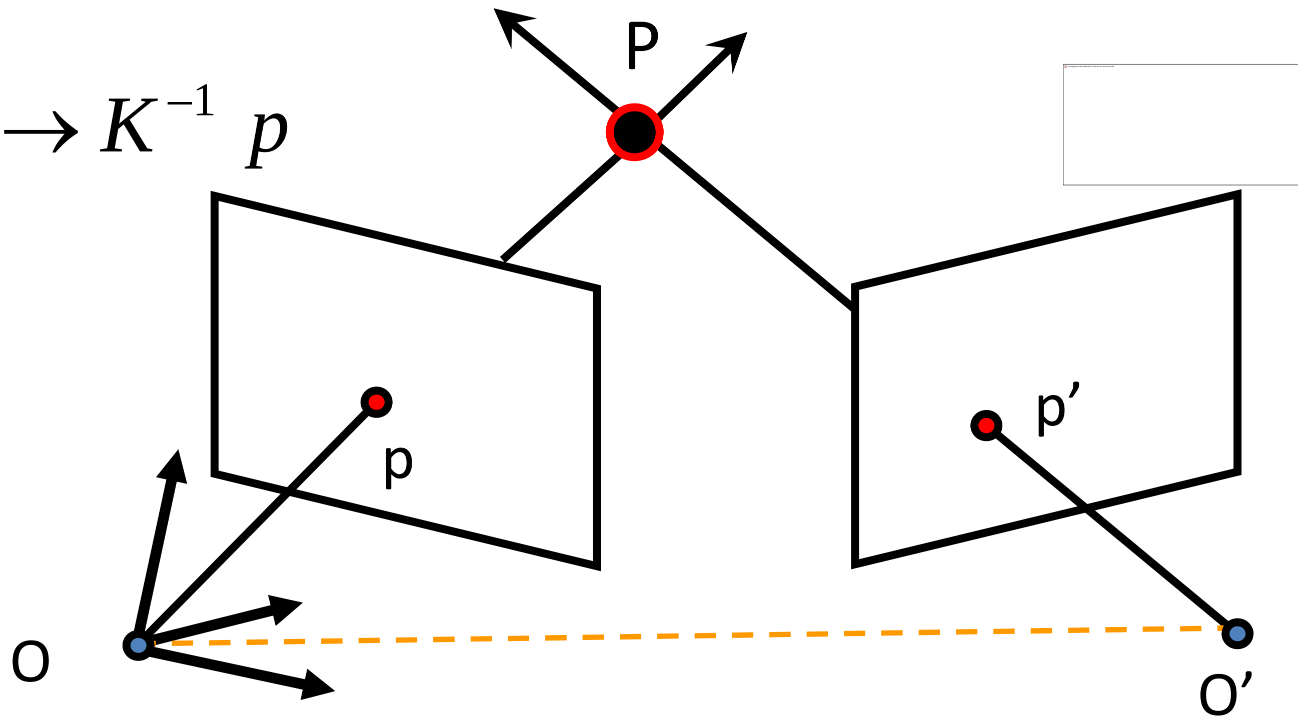


$$P \rightarrow M\,P \;\rightarrow p = \begin{bmatrix} u \\ v \end{bmatrix} \qquad\qquad M = \boxed{K}\begin{bmatrix} I & 0 \end{bmatrix}$$

unknown

Triangulation

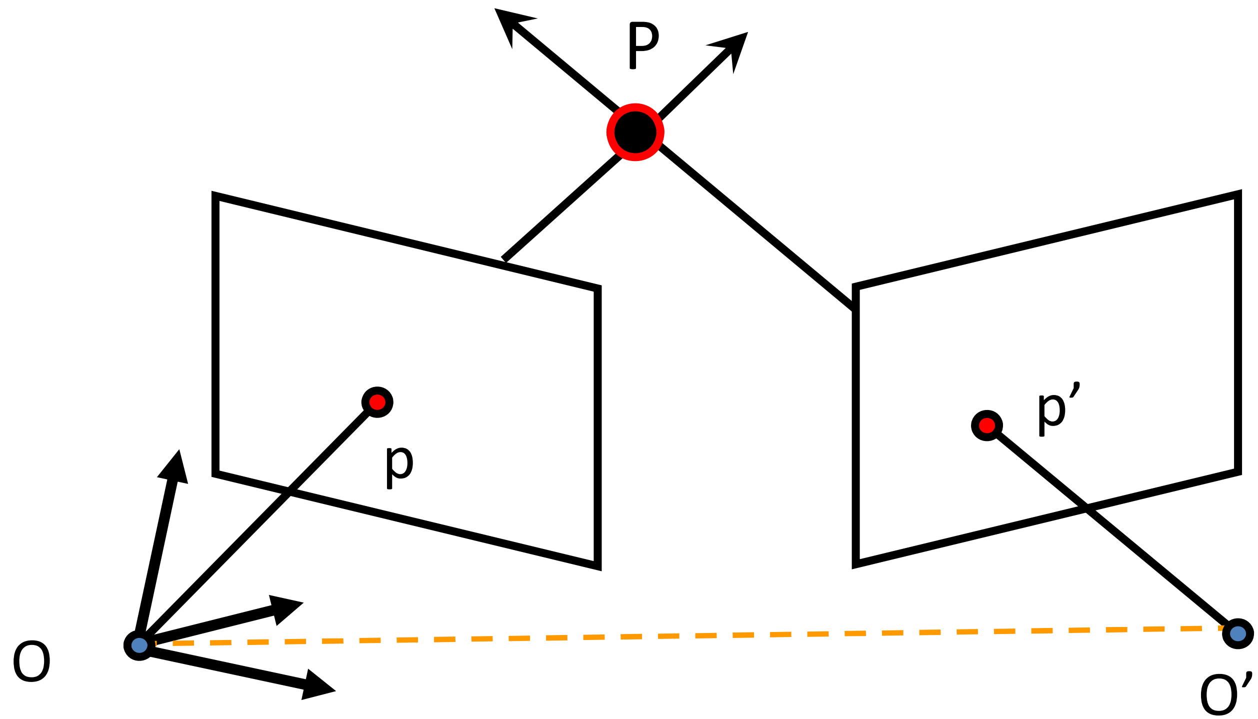
$$p \to K^{-1}\ p$$



$$p^T \cdot [T_\times] \cdot R\ p' = 0 \to (K^{-1}\ p)^T \cdot [T_\times] \cdot R\ K'^{-1}\ p' = 0$$

$$p^T\ \boxed{K^{-T} \cdot [T_\times] \cdot R\ K'^{-1}}\ p' = 0 \to p^T\ \boxed{F}\ p' = 0$$

Triangulation

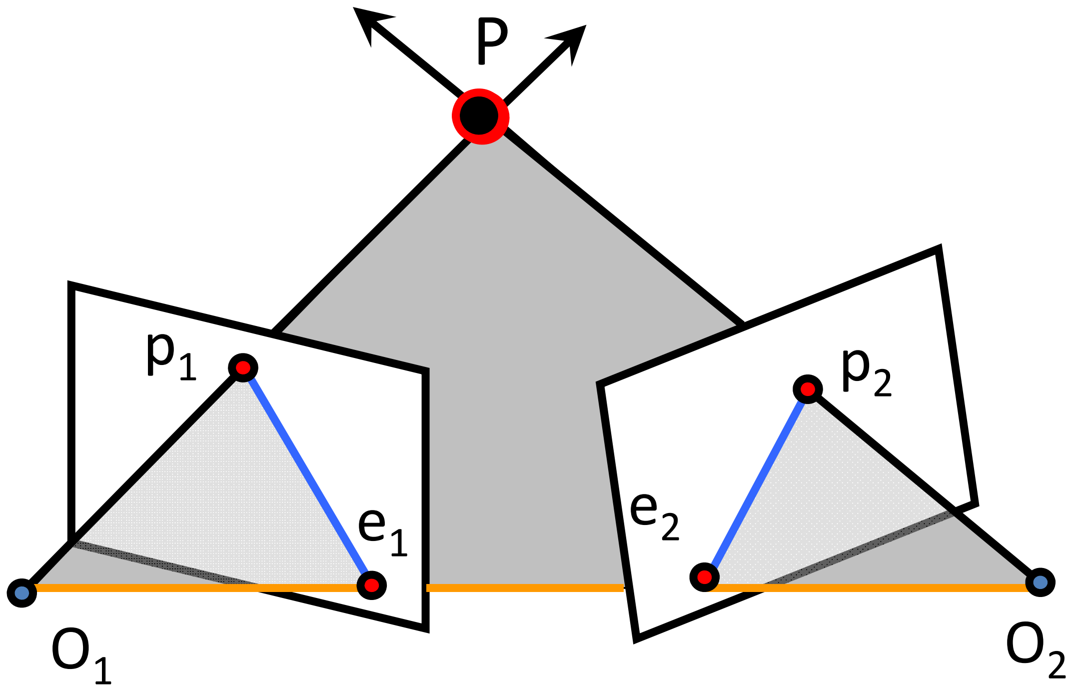


$$\boxed{p^T \, F \, p' = 0}$$

F = Fundamental Matrix

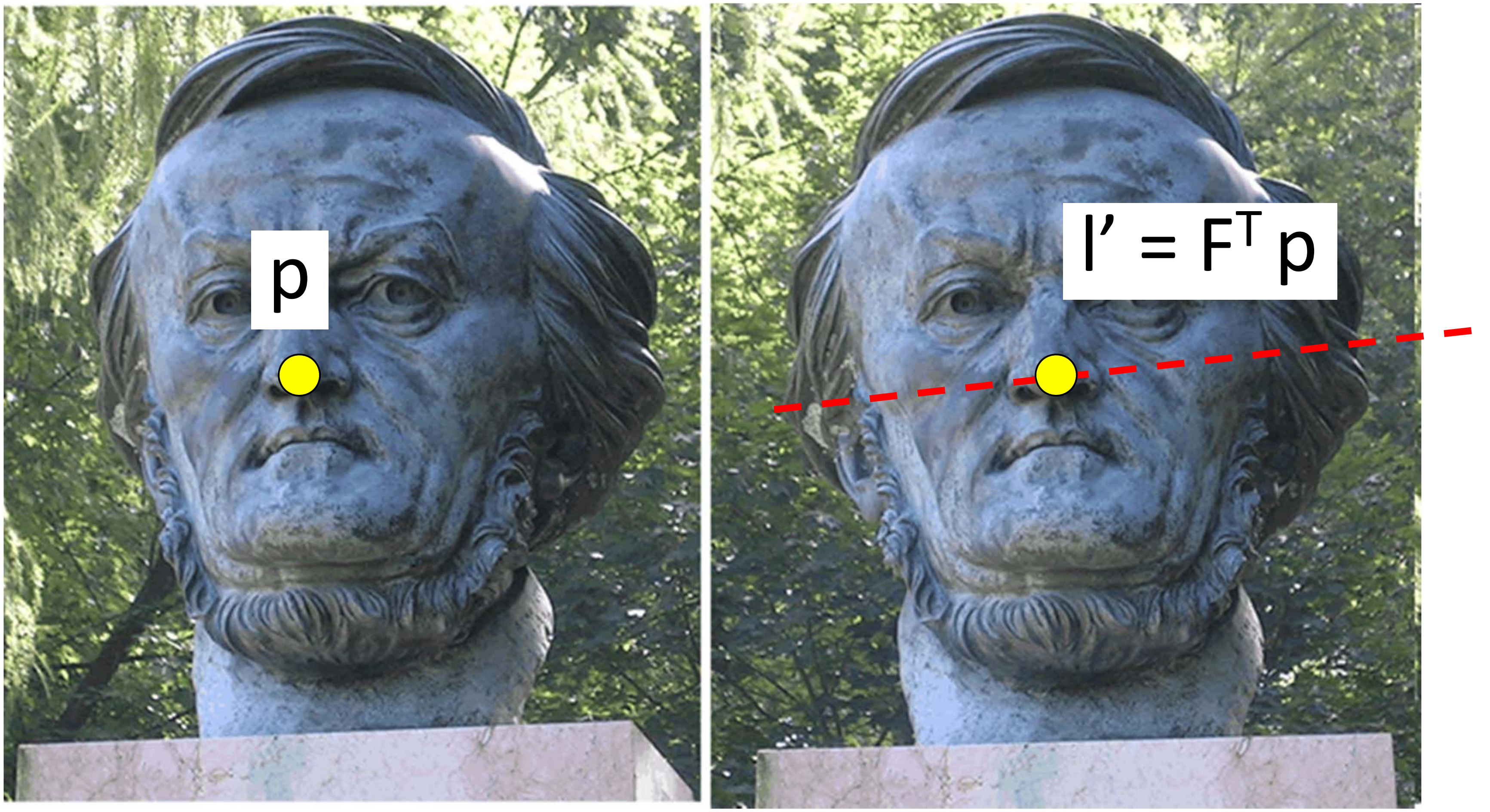
(Faugeras and Luong, 1992)

Triangulation



- $F\,p_2$ is the epipolar line associated with p_2 ($l_1 = F\,p_2$)
- $F^T p_1$ is the epipolar line associated with p_1 ($l_2 = F^T p_1$)
- F is singular (rank two)
- $F\,e_2 = 0$ and $F^T e_1 = 0$
- F is 3x3 matrix; 7 DOF

Why is F useful?



- Suppose F is known
- No additional information about the scene and camera is given
- Given a point on left image, how can I find the corresponding point on right image?

Why is F useful?

- F captures information about the epipolar geometry of 2 views + camera parameters

- MORE IMPORTANTLY: F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)

- Powerful tool in:
  - 3D reconstruction
  - Multi-view object/scene matching

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Reading:
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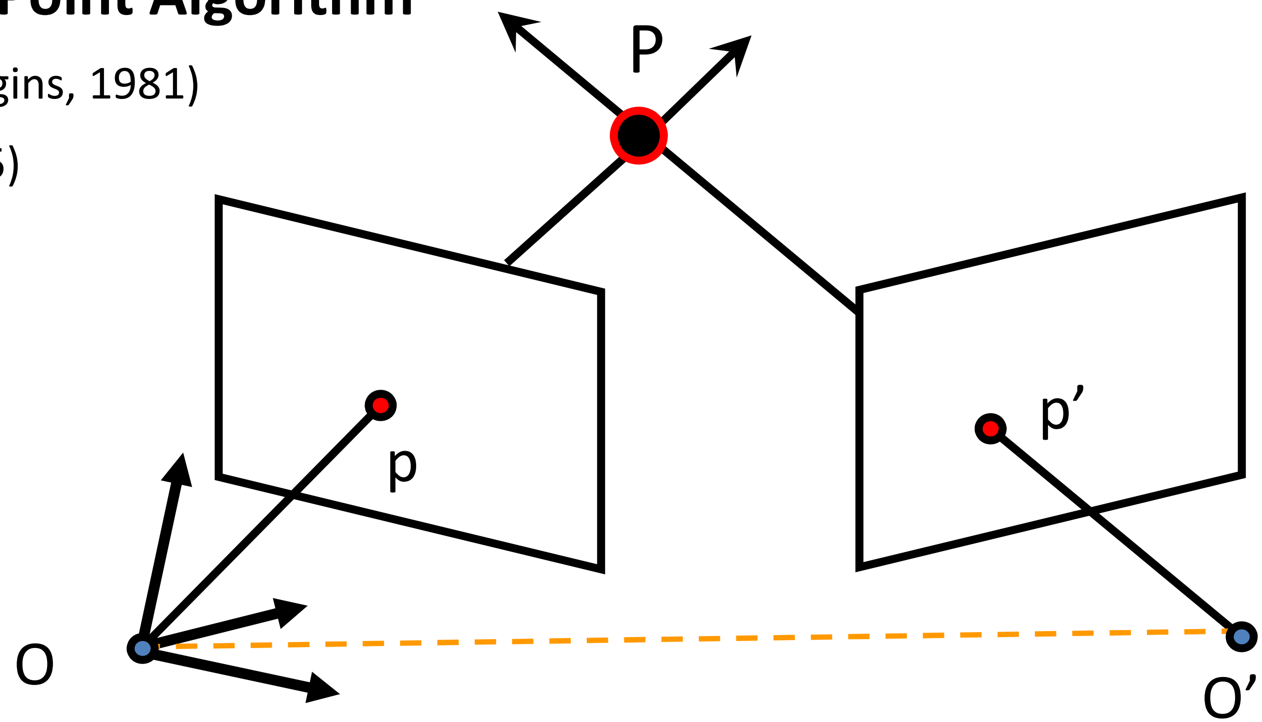
Estimating F

The Eight-Point Algorithm

(Longuet-Higgins, 1981)

(Hartley, 1995)



$$P \rightarrow p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \qquad P \rightarrow p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} \qquad p^T F p' = 0$$

Estimating F

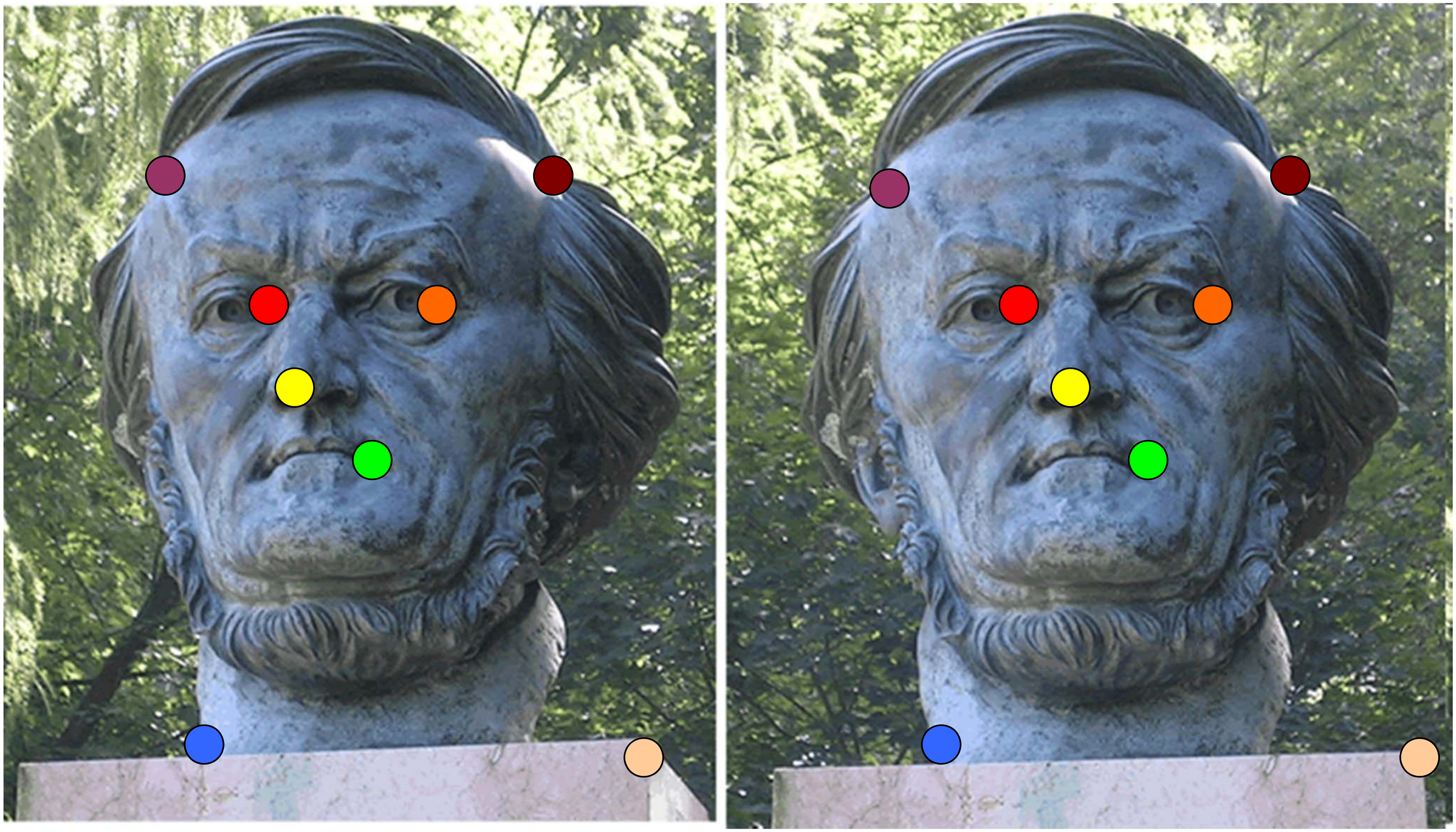
$$p^T F p' = 0 \quad \Rightarrow$$

$$(u, v, 1) \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad (uu', uv', u, vu', vv', v, u', v', 1) \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

Let's take 8 corresponding points

Estimating F



Estimating F

$$\mathbf{W}\;\begin{pmatrix} u_1u_1' & u_1v_1' & u_1 & v_1u_1' & v_1v_1' & v_1 & u_1' & v_1' & 1 \\ u_2u_2' & u_2v_2' & u_2 & v_2u_2' & v_2v_2' & v_2 & u_2' & v_2' & 1 \\ u_3u_3' & u_3v_3' & u_3 & v_3u_3' & v_3v_3' & v_3 & u_3' & v_3' & 1 \\ u_4u_4' & u_4v_4' & u_4 & v_4u_4' & v_4v_4' & v_4 & u_4' & v_4' & 1 \\ u_5u_5' & u_5v_5' & u_5 & v_5u_5' & v_5v_5' & v_5 & u_5' & v_5' & 1 \\ u_6u_6' & u_6v_6' & u_6 & v_6u_6' & v_6v_6' & v_6 & u_6' & v_6' & 1 \\ u_7u_7' & u_7v_7' & u_7 & v_7u_7' & v_7v_7' & v_7 & u_7' & v_7' & 1 \\ u_8u_8' & u_8v_8' & u_8 & v_8u_8' & v_8v_8' & v_8 & u_8' & v_8' & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0 \quad \mathbf{f}$$

- Homogeneous system $\quad \mathbf{W}\,\mathbf{f} = 0$
- Rank 8 $\longrightarrow$ A non-zero solution exists (unique)
- If N>8 $\longrightarrow$ Lsq. solution by SVD! $\longrightarrow \hat{\mathrm{F}}$ $\quad \|\mathbf{f}\| = 1$

Estimating F

$$p^T \hat{F} p' = 0$$

The estimated $\hat{F}$ may have full rank ($\det(\hat{F}) \neq 0$)
(F should have rank=2 instead)

Find F that minimizes $$\left\| F - \hat{F} \right\| = 0$$

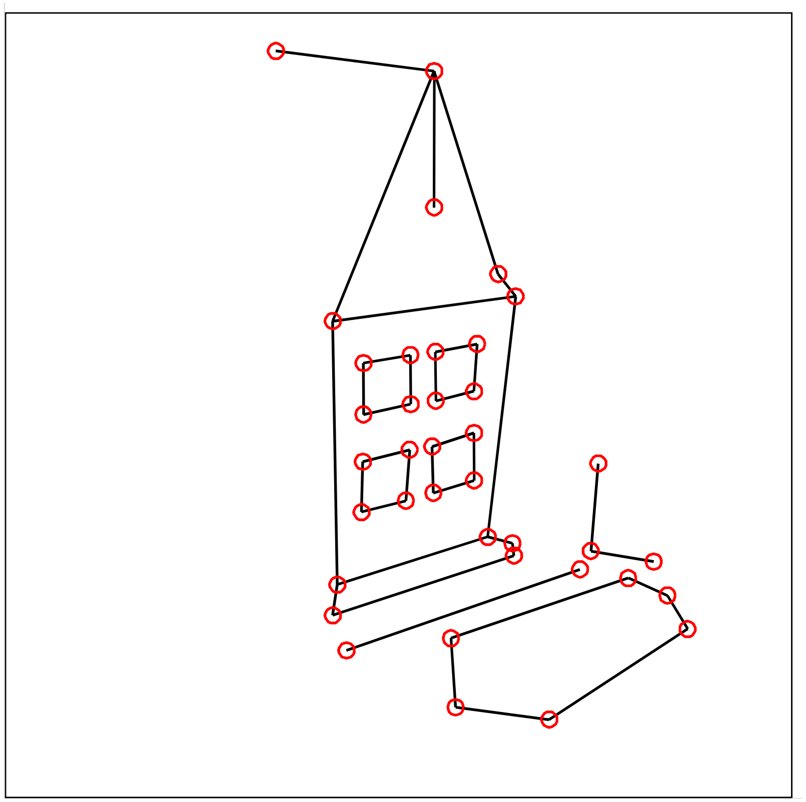
Frobenius norm (*)

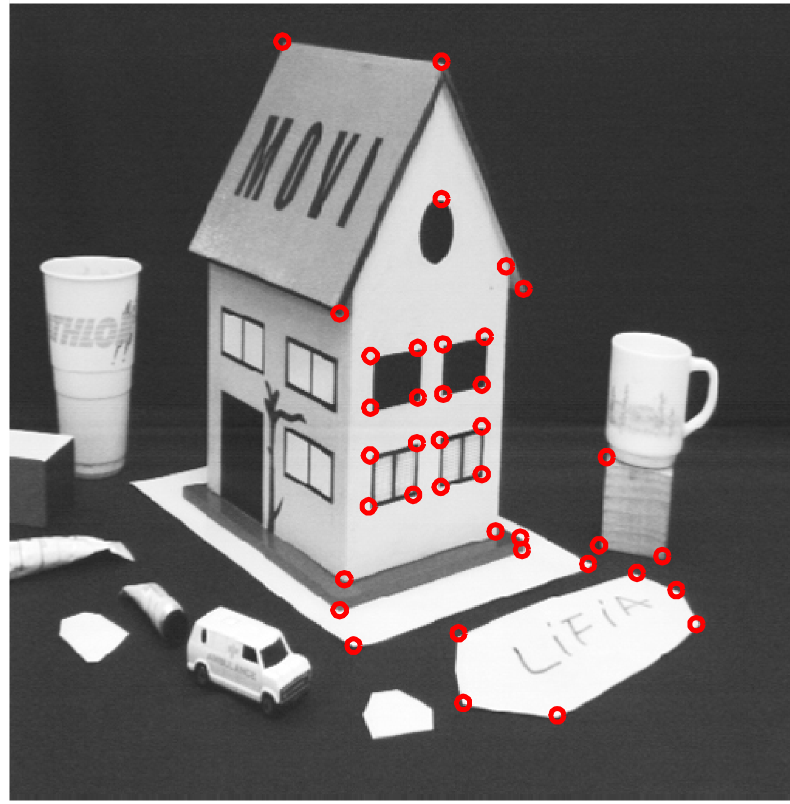
Subject to det(F)=0

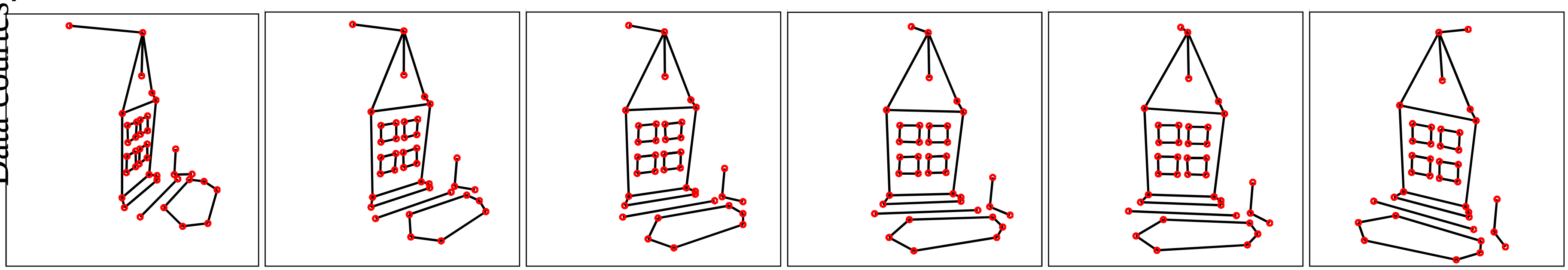
SVD (again!) can be used to solve this problem

(*) Sqrt root of the sum pf squares of all entries

Example

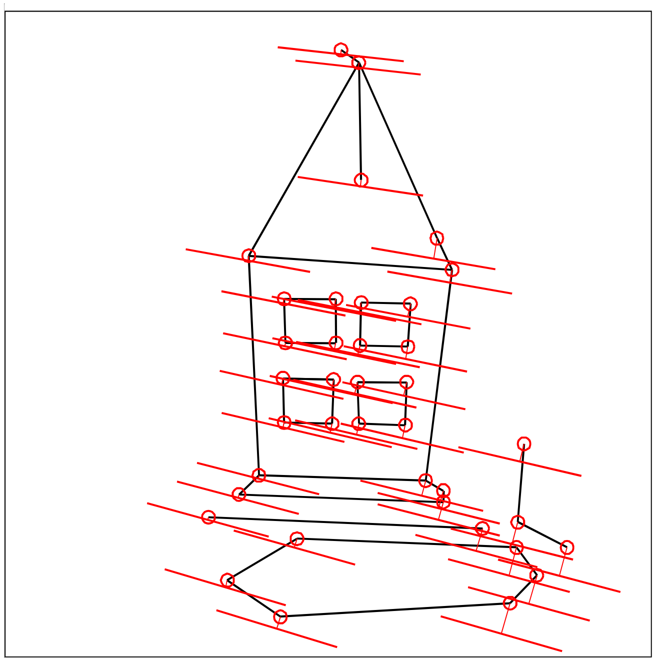


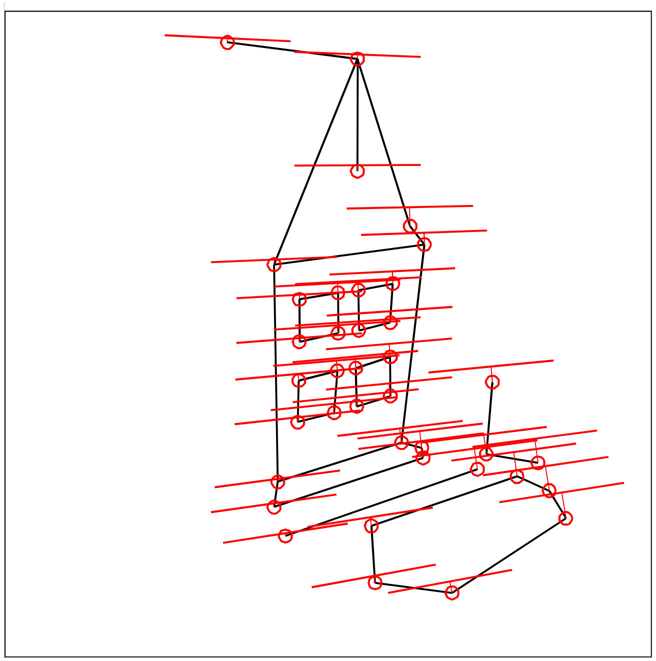




Data courtesy of R. Mohr and B. Boufama.

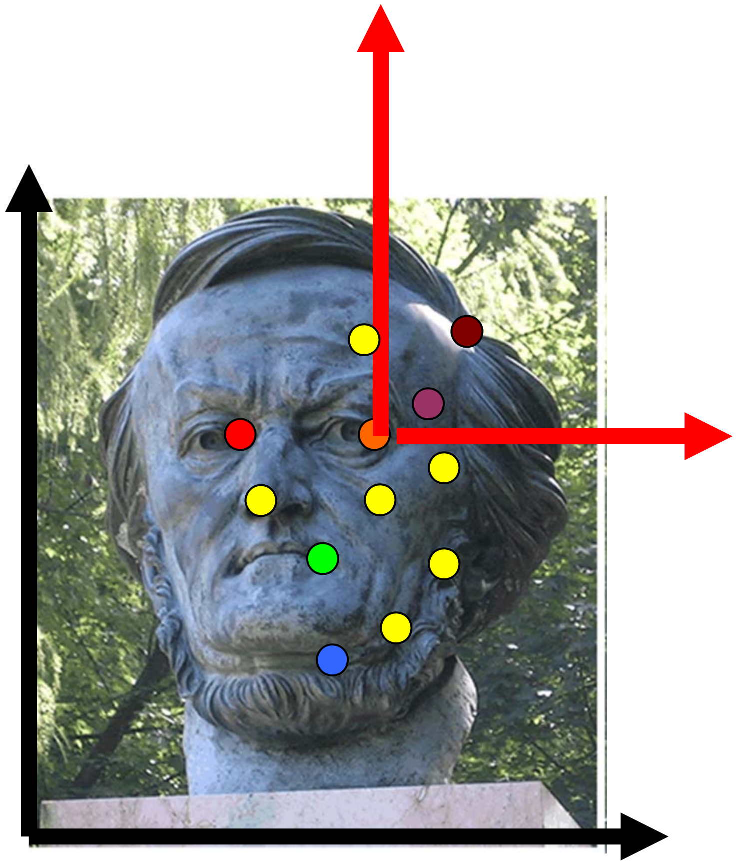
Example





Mean errors:
10.0pixel
9.1pixel

Normalization



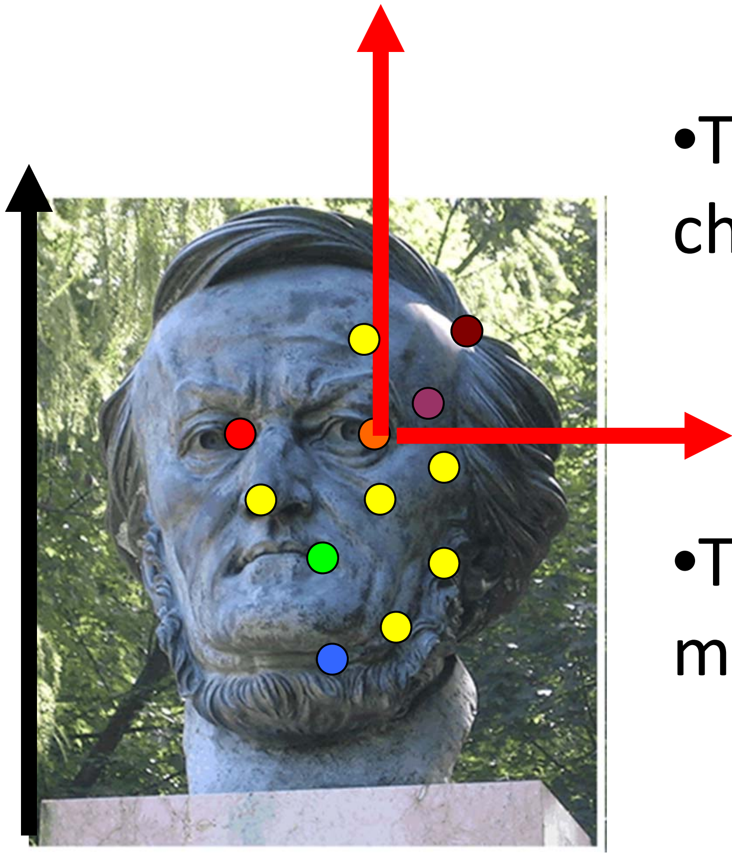
Is the accuracy in estimating F function of the ref. system in the image plane?

E.g. under similarity transformation (T = scale + translation):

$$q_i = T_i \, p_i \qquad q'_i = T'_i \, p'_i$$

Does the accuracy in estimating F change if a transformation T is applied?

Normalization

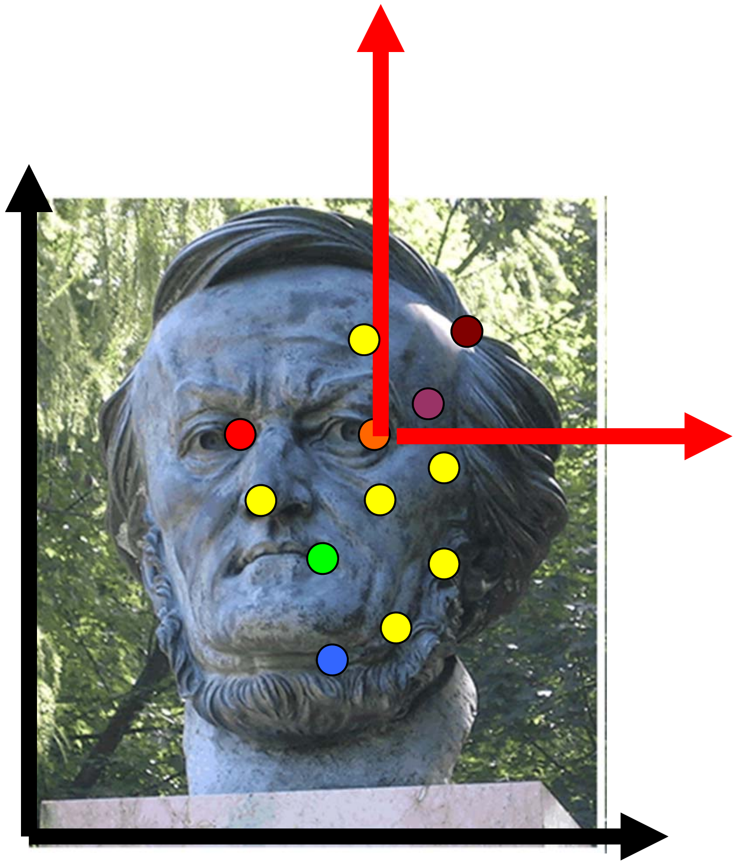


•The accuracy in estimating F does change if a transformation T is applied

•There exists a T for which accuracy is maximized

Why?

Normalization



$$\mathbf{W}\,\mathbf{f} = 0, \quad \|\mathbf{f}\| = 1 \xrightarrow{\text{Lsq solution by SVD}} \mathrm{F}$$

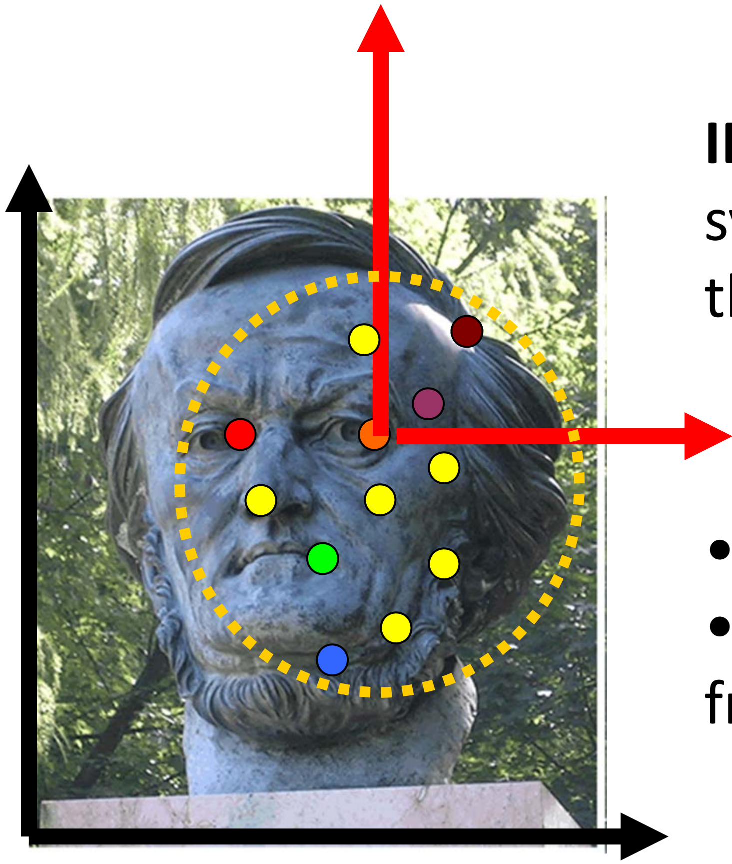
- SVD enforces Rank(W)=8

- Recall the structure of W:
  Highly un-balance
  (not well conditioned)

-Values of W must have similar magnitude

More details HZ pag 108

Normalization



IDEA: Transform image coordinate system (T = translation + scaling) such that:

- Origin = centroid of image points
- Mean square distance of the data points from origin is 2 pixels

$$q_i = T_i \; p_i \qquad q'_i = T'_i \; p'_i$$

(normalization)

The Normalized Eight-Point Algorithm

0. Compute T_i and T_i'

1. Normalize coordinates:

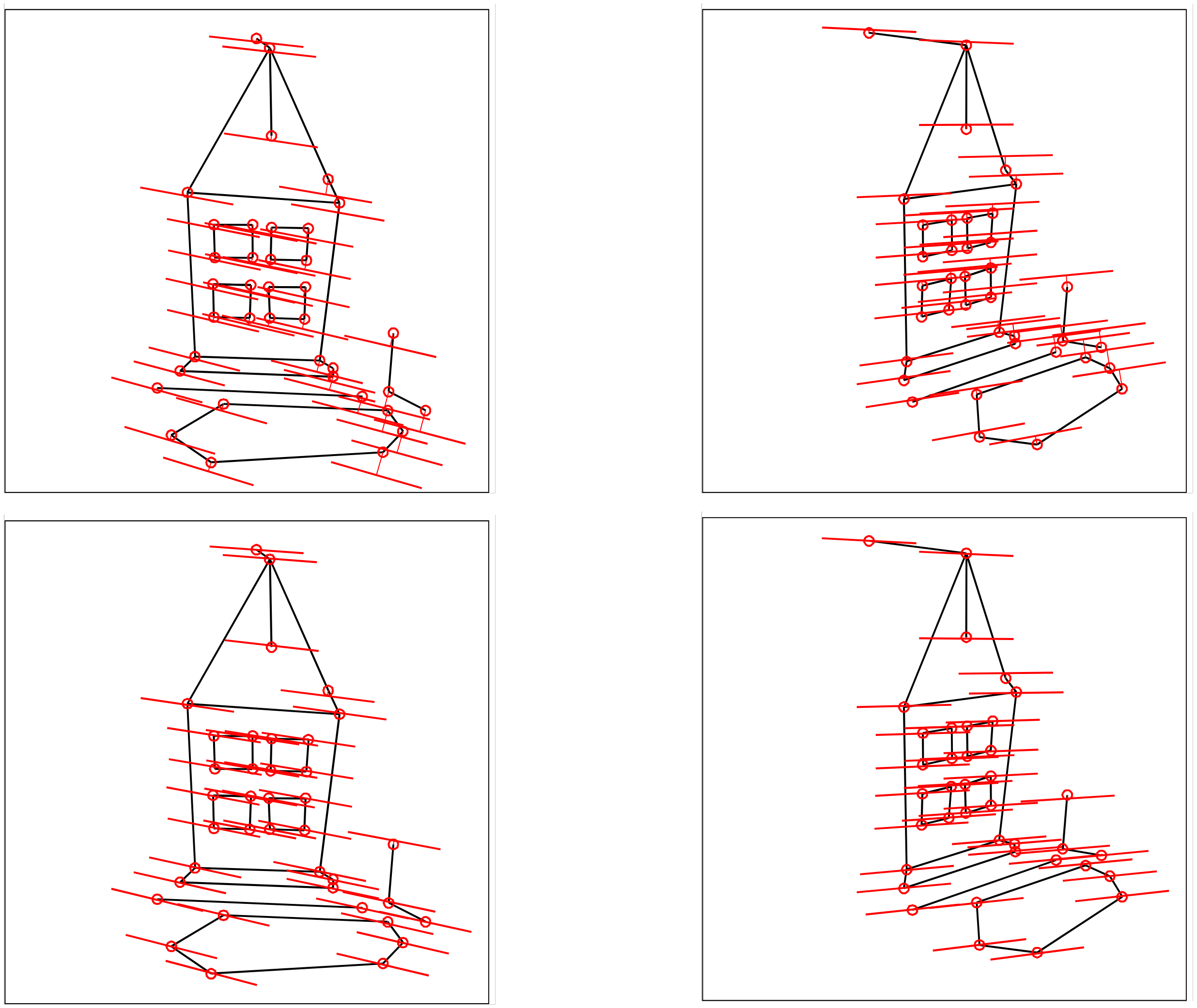
$$q_i = T_i \; p_i \qquad\qquad q_i' = T_i' \, p_i'$$

2. Use the eight-point algorithm to compute F'_q from the points q_i and q_i' .

3. Enforce the rank-2 constraint. $\rightarrow F_q$ $\begin{cases} q^T \; F_q \; q' = 0 \\ \det(F_q) = 0 \end{cases}$

4. De-normalize F_q: $F = T'^T F_q T$

Example



Without transformation

Mean errors:
10.0pixel
9.1pixel

With transformation

Mean errors:
1.0pixel
0.9pixel

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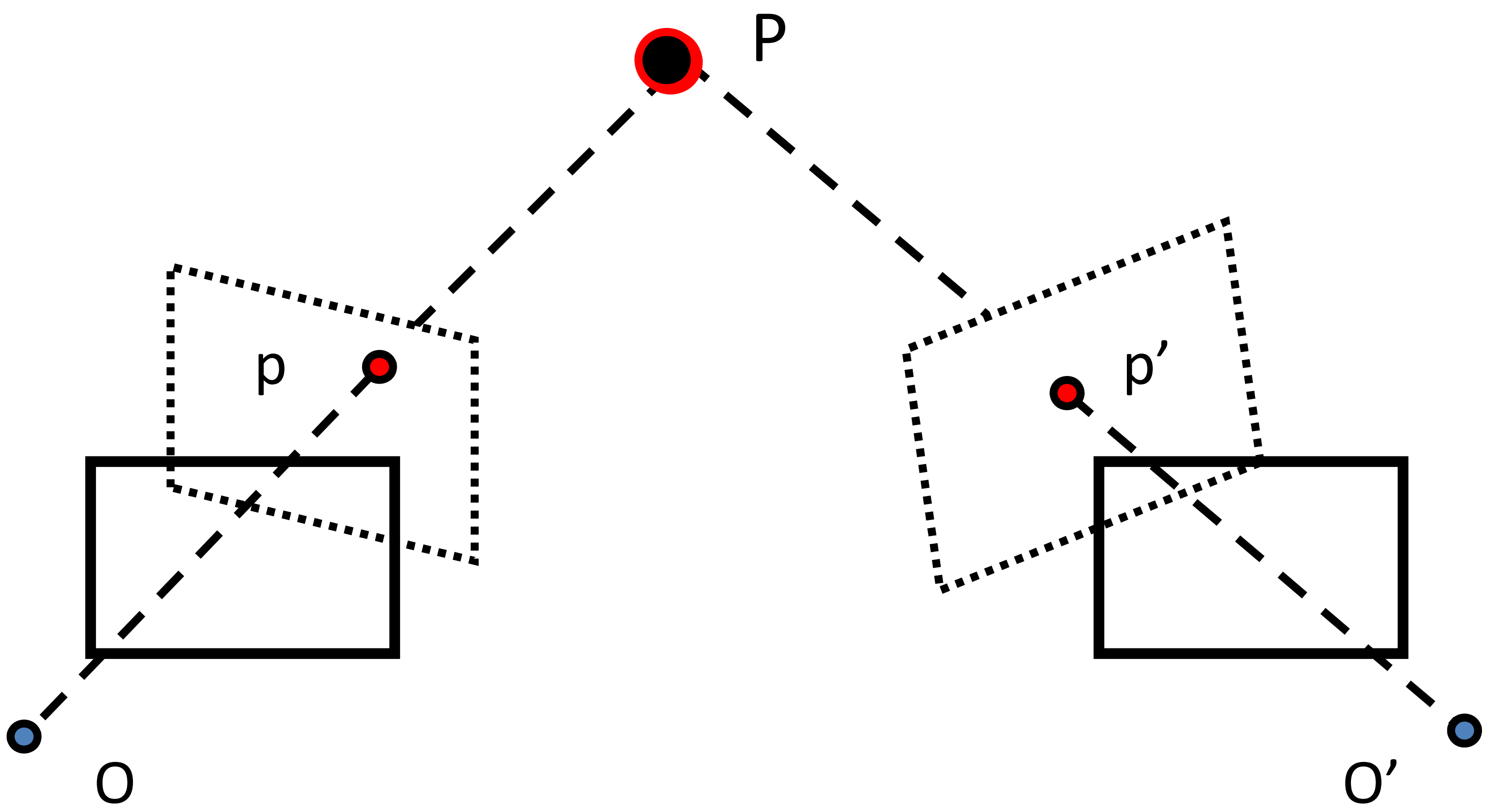
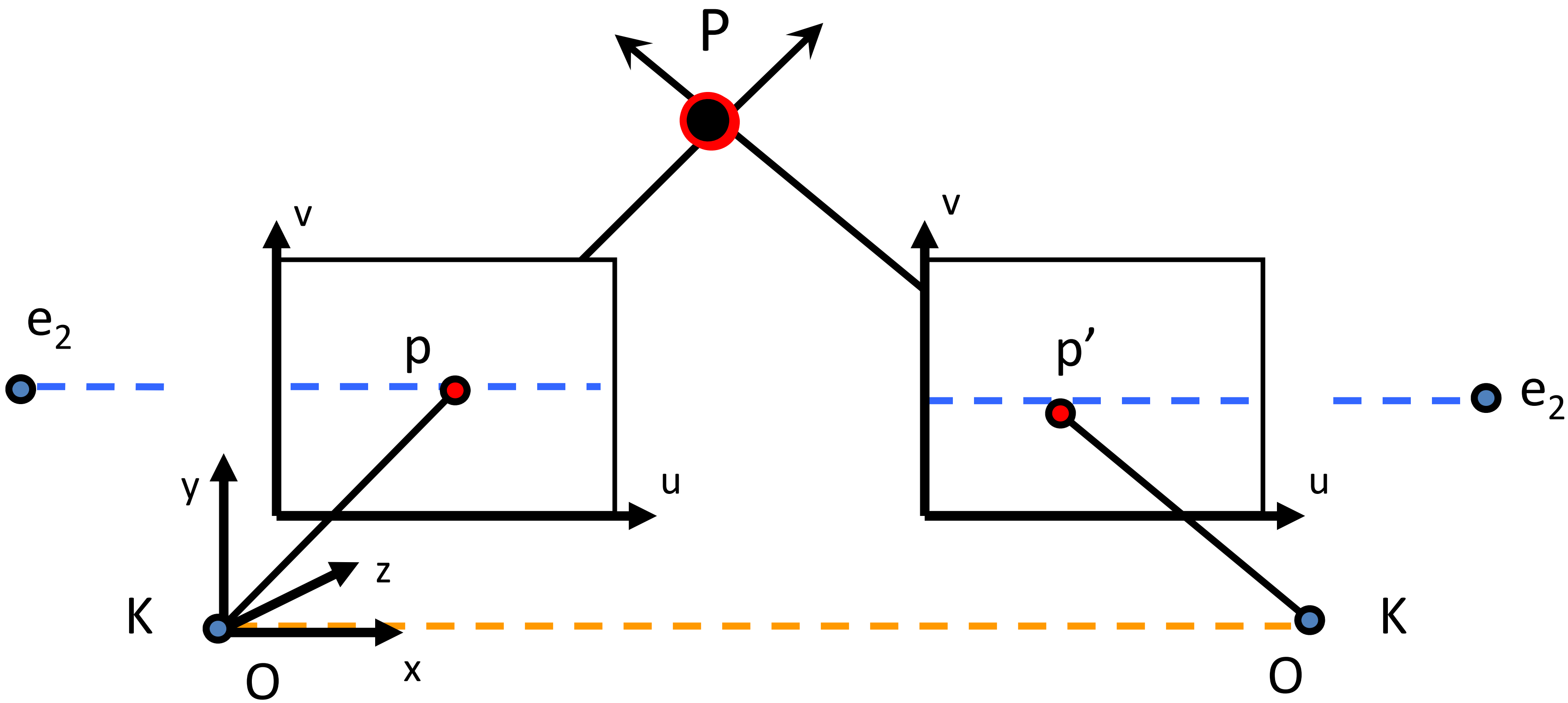
Rectification



- Make two camera images “parallel”
  - Correspondence problem becomes easier

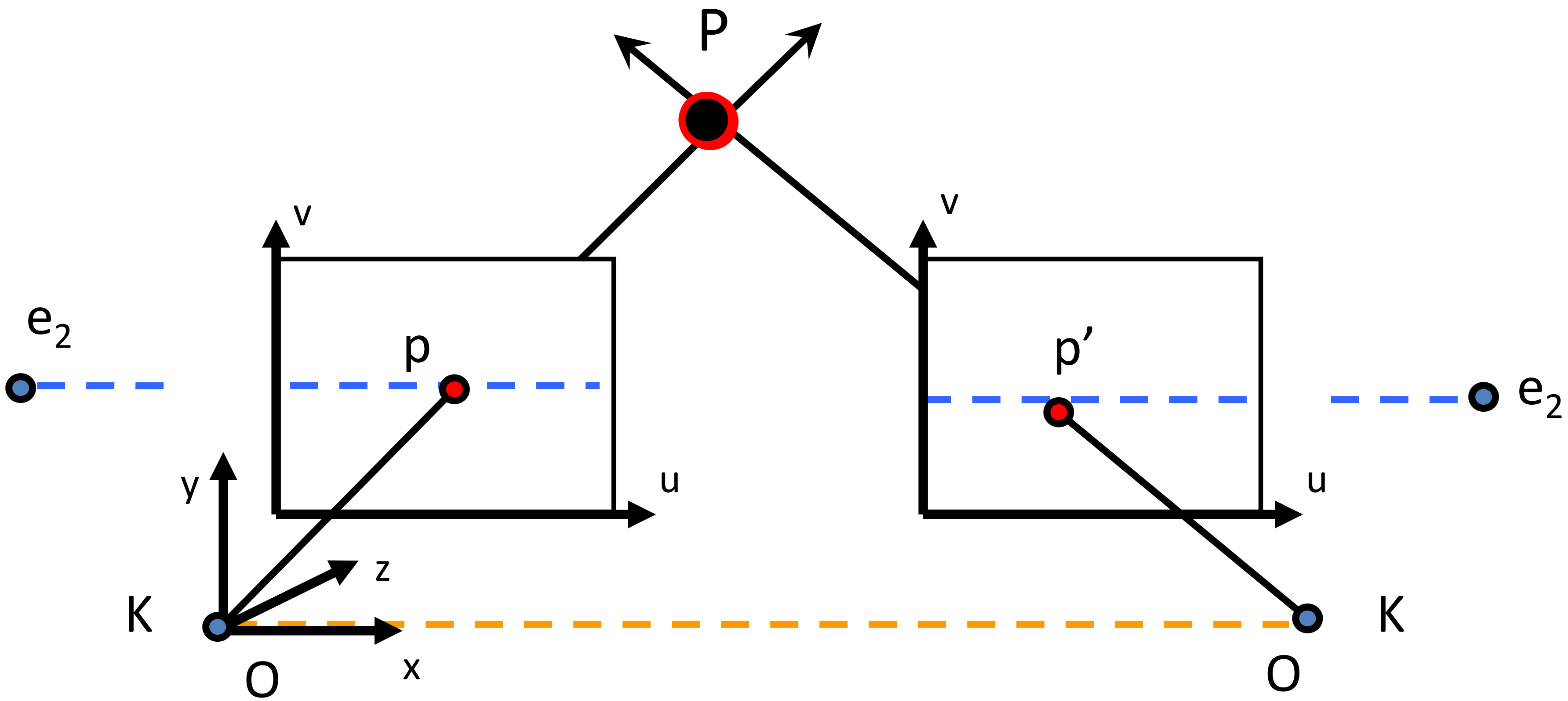
Rectification



- Parallel epipolar lines
- Epipoles at infinity
- v = v'

Let's see why....

Rectification



K_1=K_2 = known

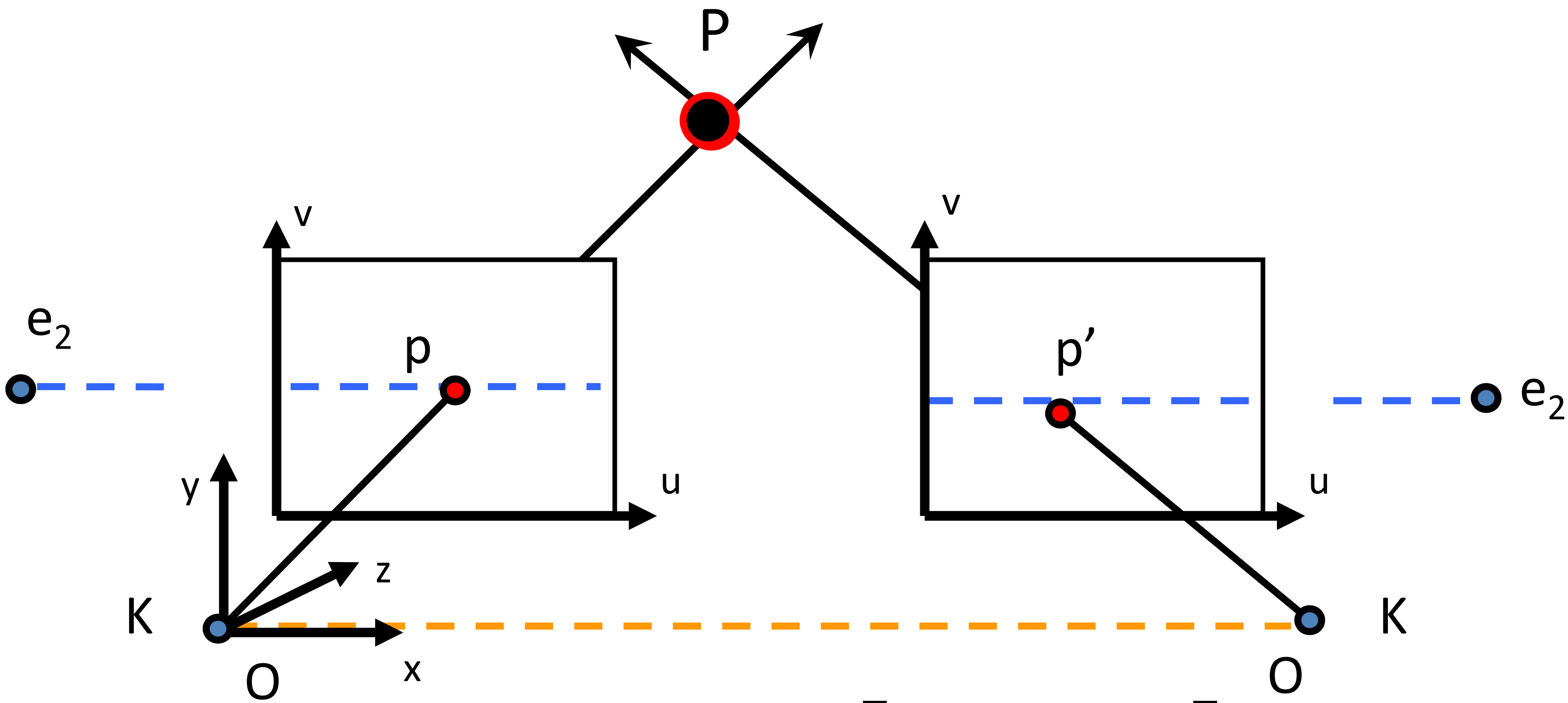
x parallel to O_1O_2

$$E = [t_{\times}]R$$

Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

Rectification



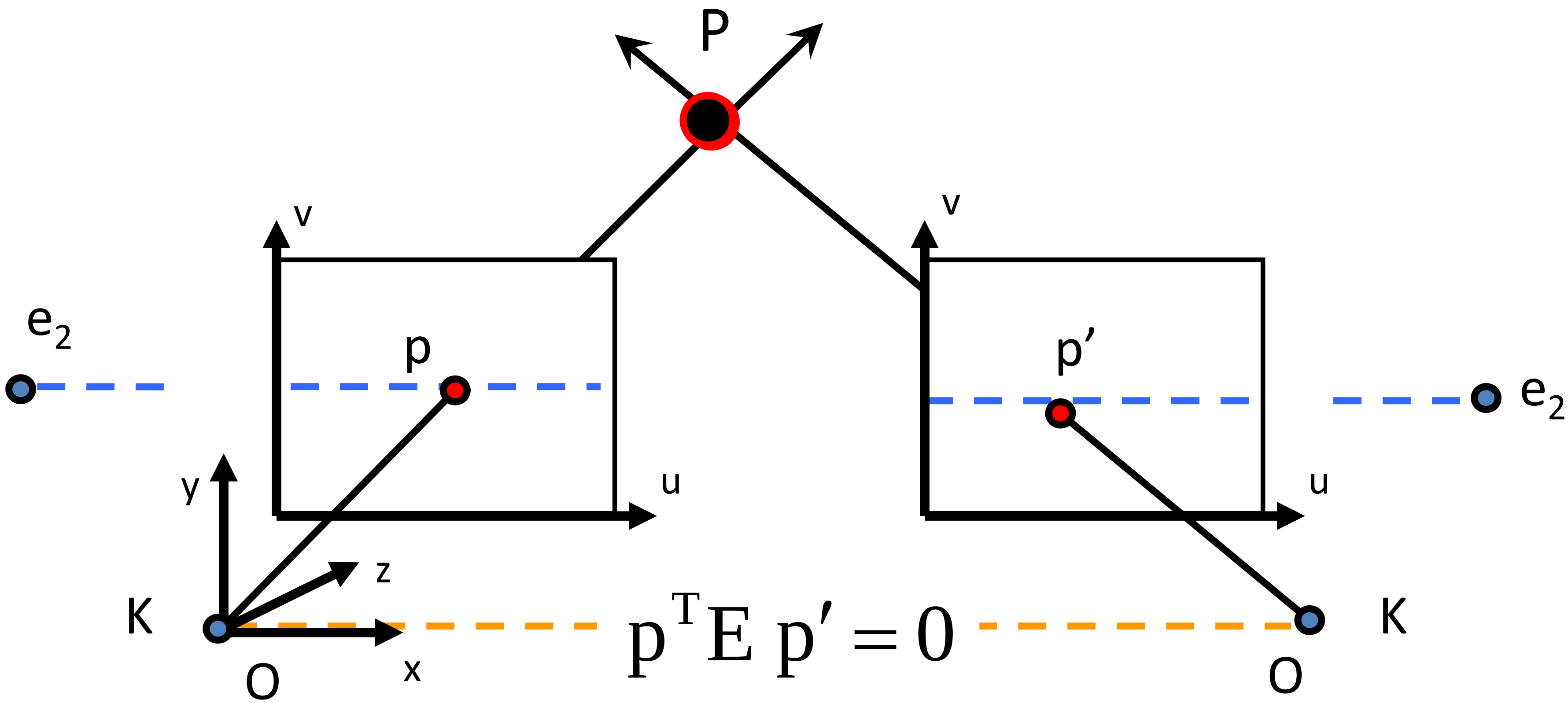
K_1=K_2 = known

x parallel to O_1O_2

$$E = [t_{\times}]R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix}$$

$\rightarrow$ v = v'?

Rectification

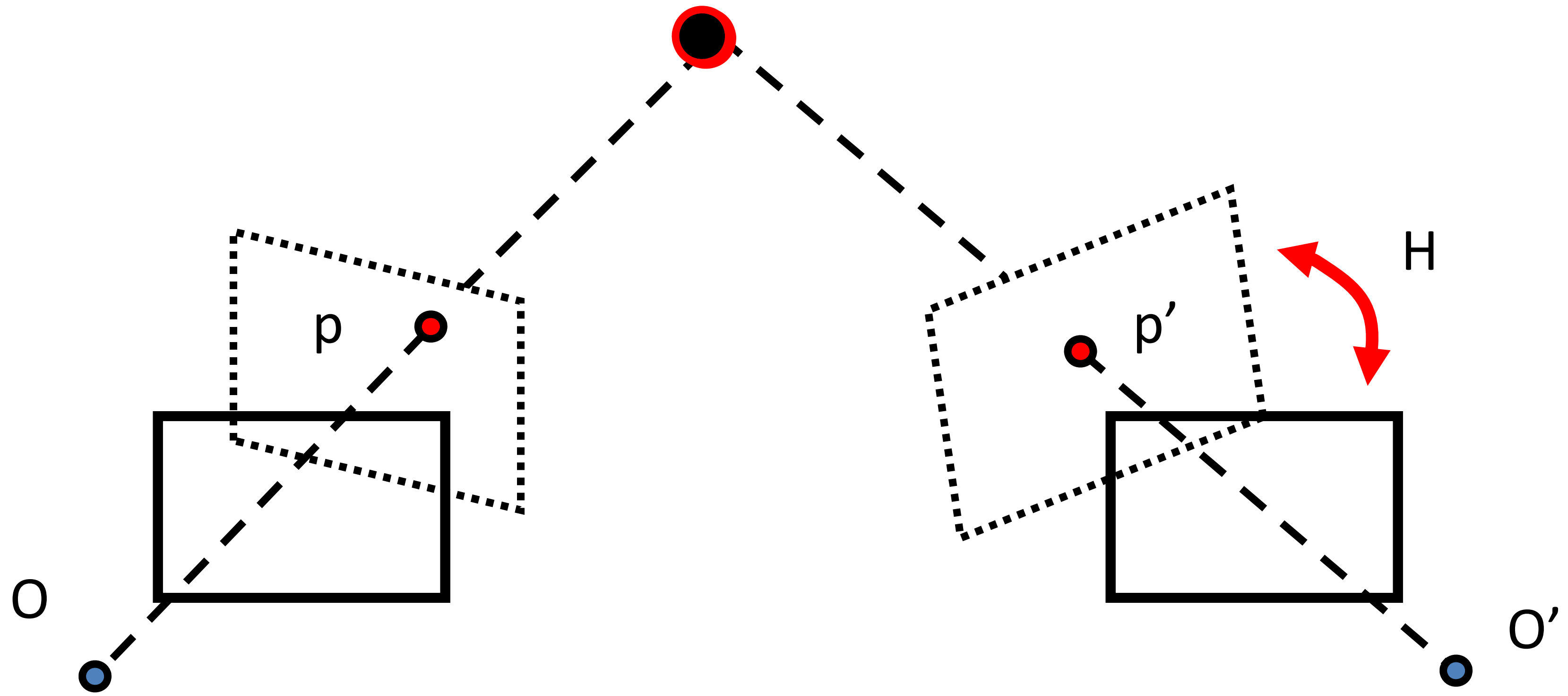


$$p^T E\, p' = 0$$

$$\begin{pmatrix} u & v & 1 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0 \qquad \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -T \\ Tv' \end{pmatrix} = 0 \qquad Tv = Tv'$$

$\rightarrow$ v = v'

Rectification



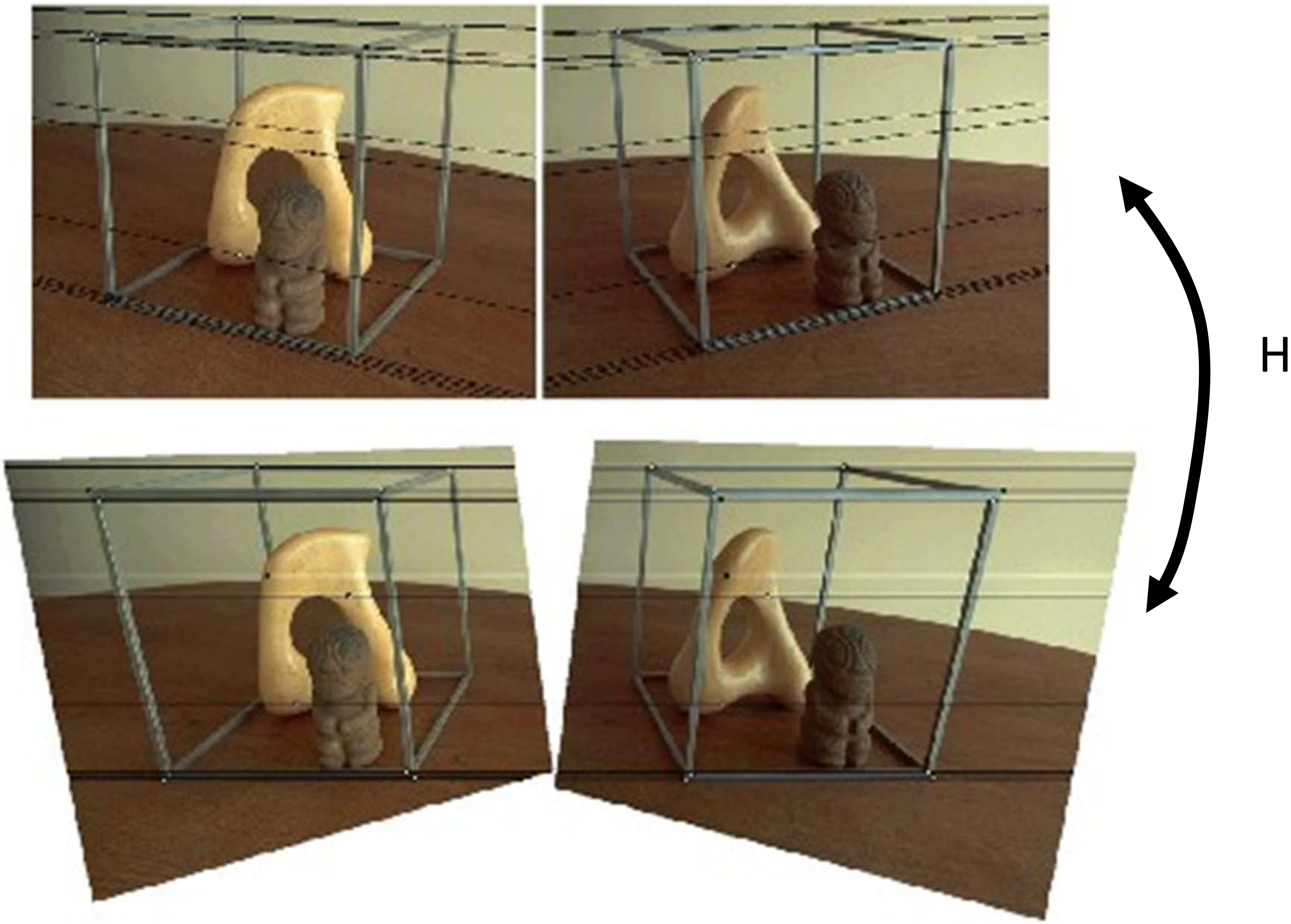
GOAL of rectification : Estimate a perspective transformation H that makes images parallel

Impose v’=v

- This leaves degrees of freedom for determining H
- If not appropriate H is chosen, severe projective distortions on image take place
- We impose a number of restriction while computing H

[HZ] Chapters: 11 (sec. 11.12)

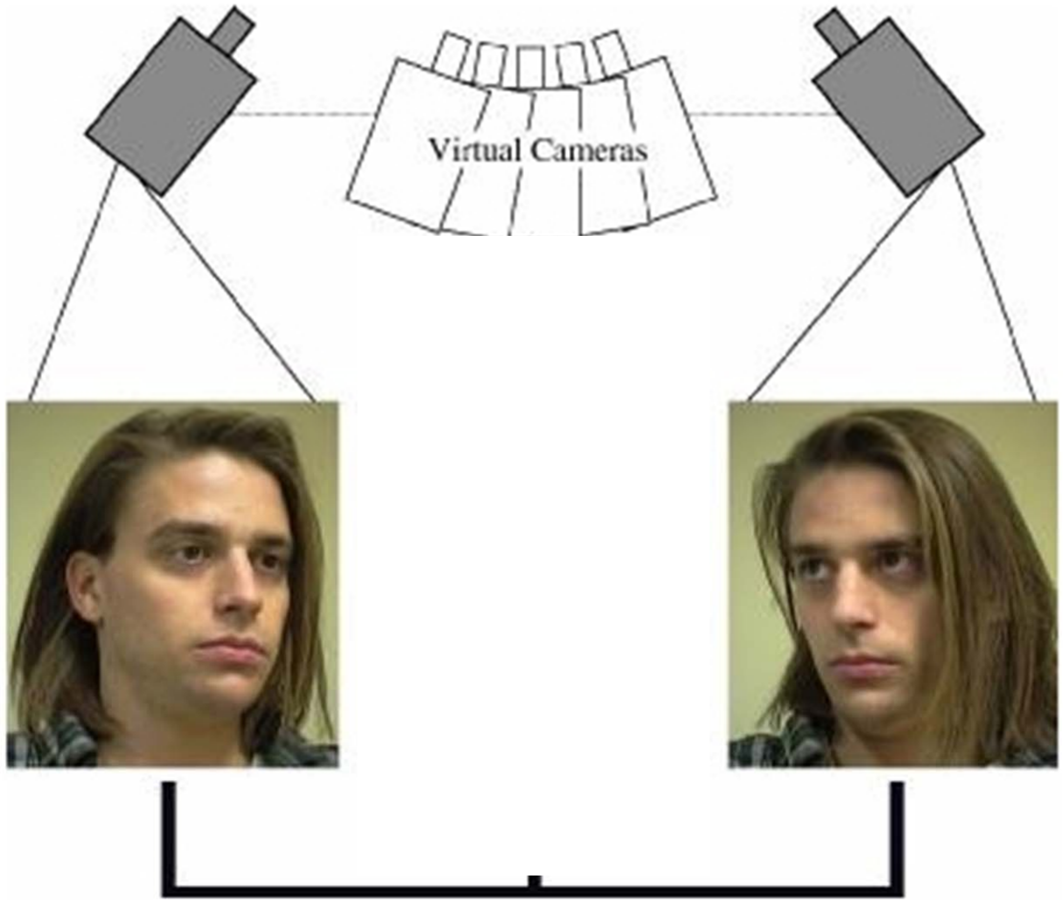
Rectification



Courtesy figure S. Lazebnik

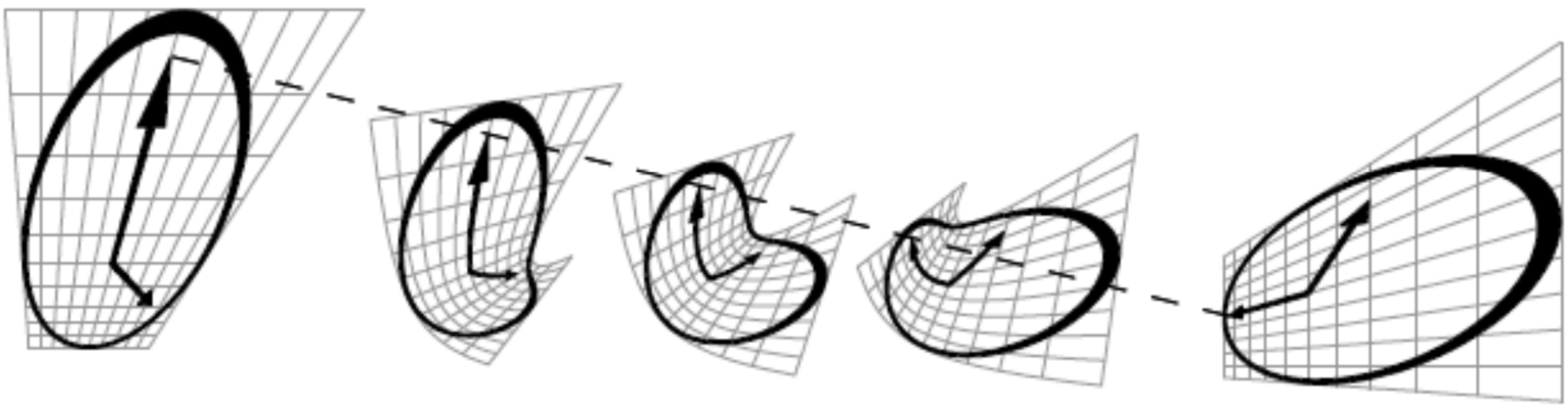
Application: view morphing

S. M. Seitz and C. R. Dyer, *Proc. SIGGRAPH 96*, 1996, 21-30

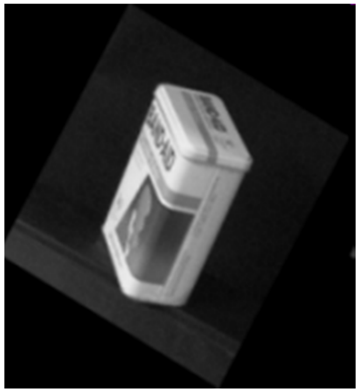


Application: view morphing

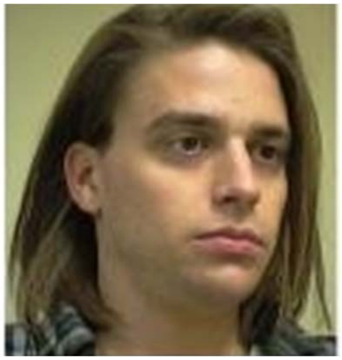
If rectification is not applied, the morphing procedure does not generate geometrically correct interpolations

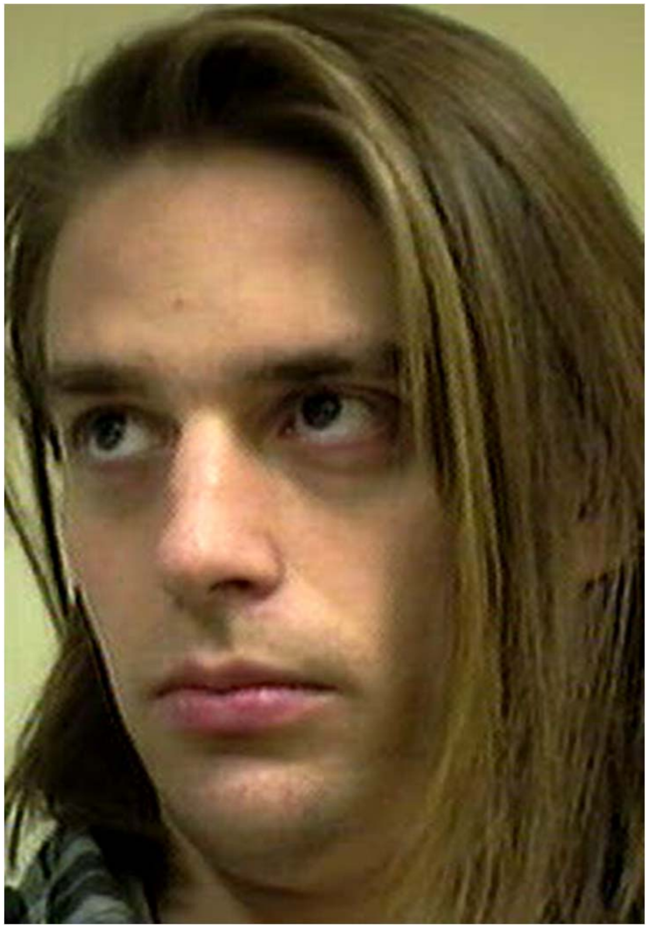


Application: view morphing

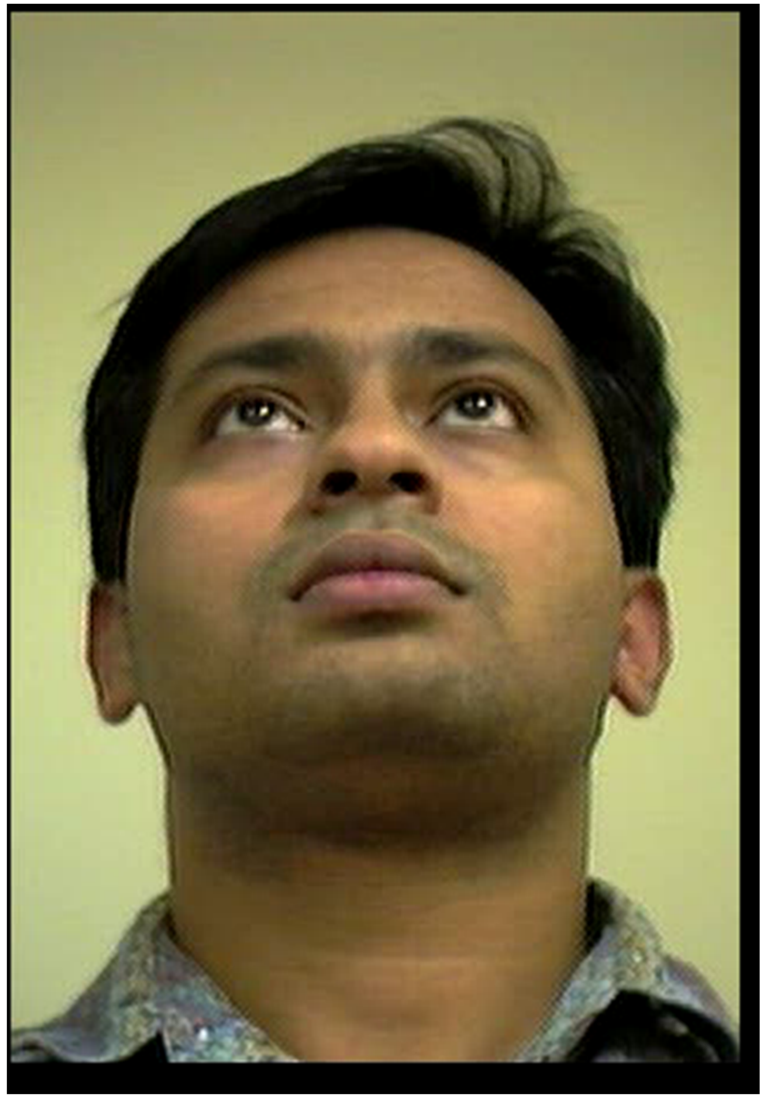


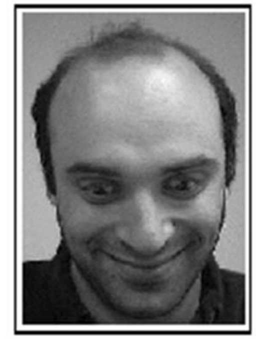
Application: view morphing



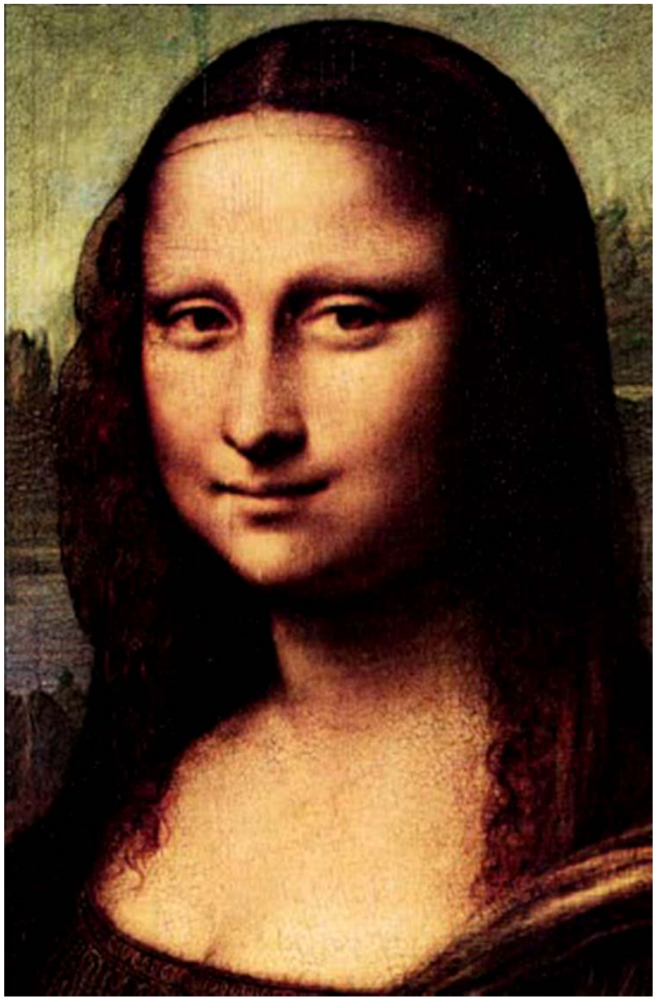


Application: view morphing





Application: view morphing



The Fundamental Matrix Song

http://danielwedge.com/fmatrix/

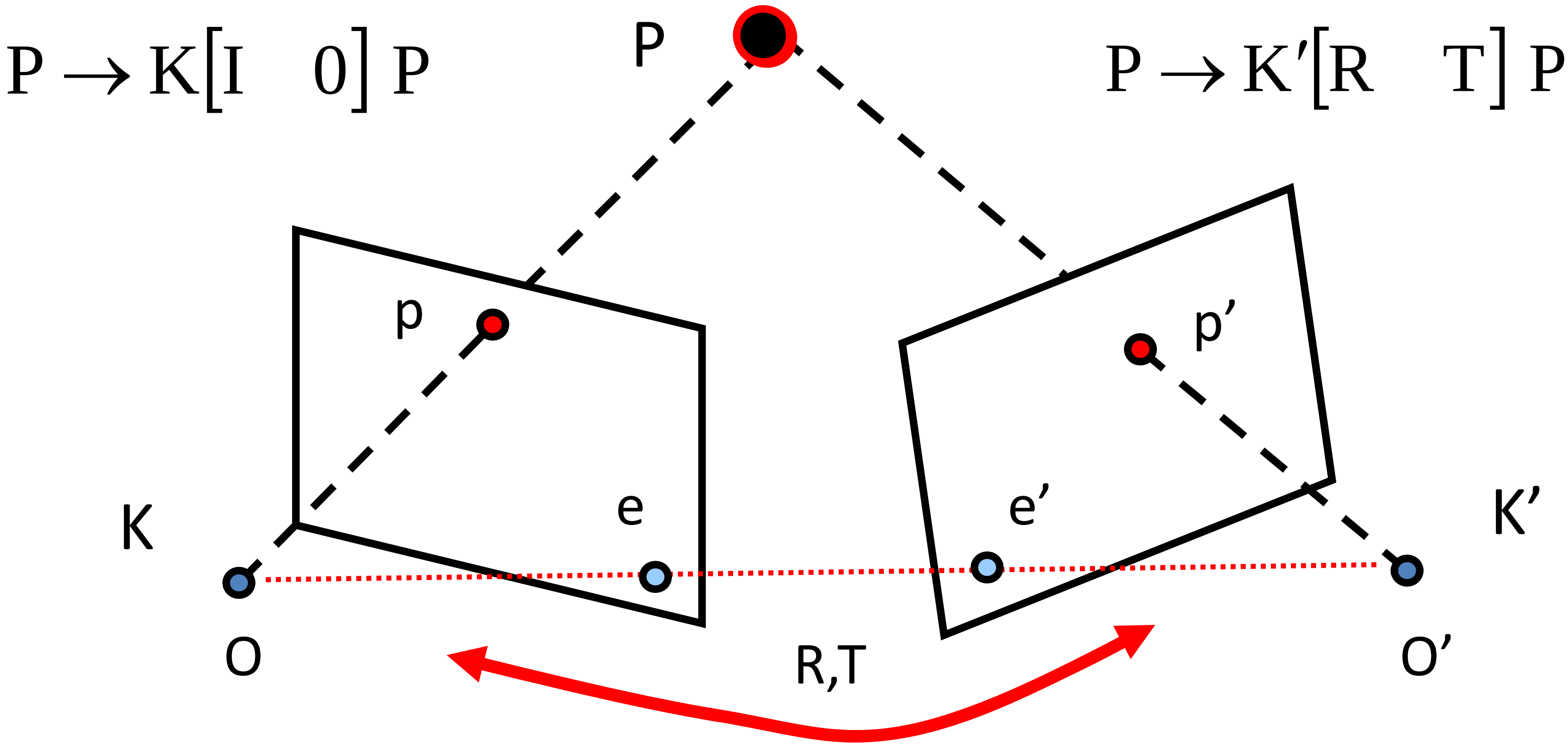
What we have learned today?

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F (Problem Set 2 (Q2))
- Rectification

Reading:
[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10

Supplementary materials

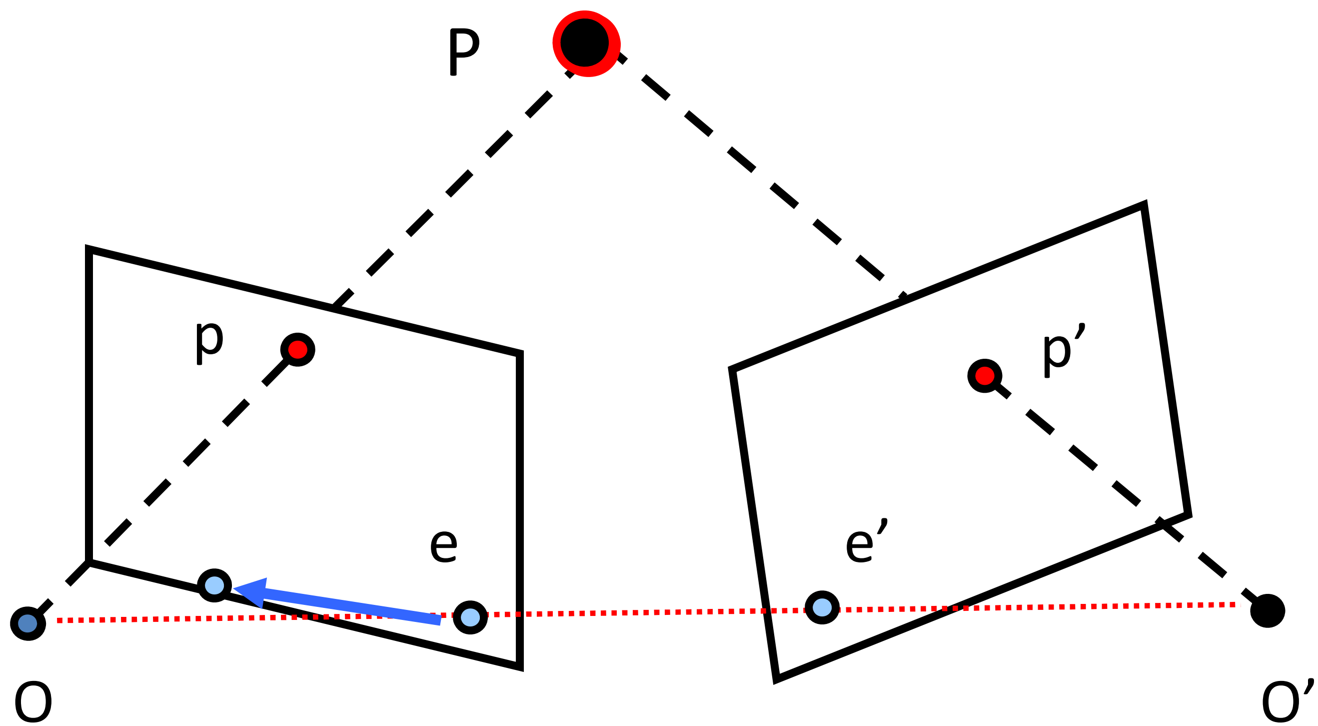
Making image planes parallel



0. Compute epipoles

$$e = K\,R^{T}\,T = [e_1 \quad e_2 \quad 1]^{T} \qquad e' = K'\,T$$

Making image planes parallel

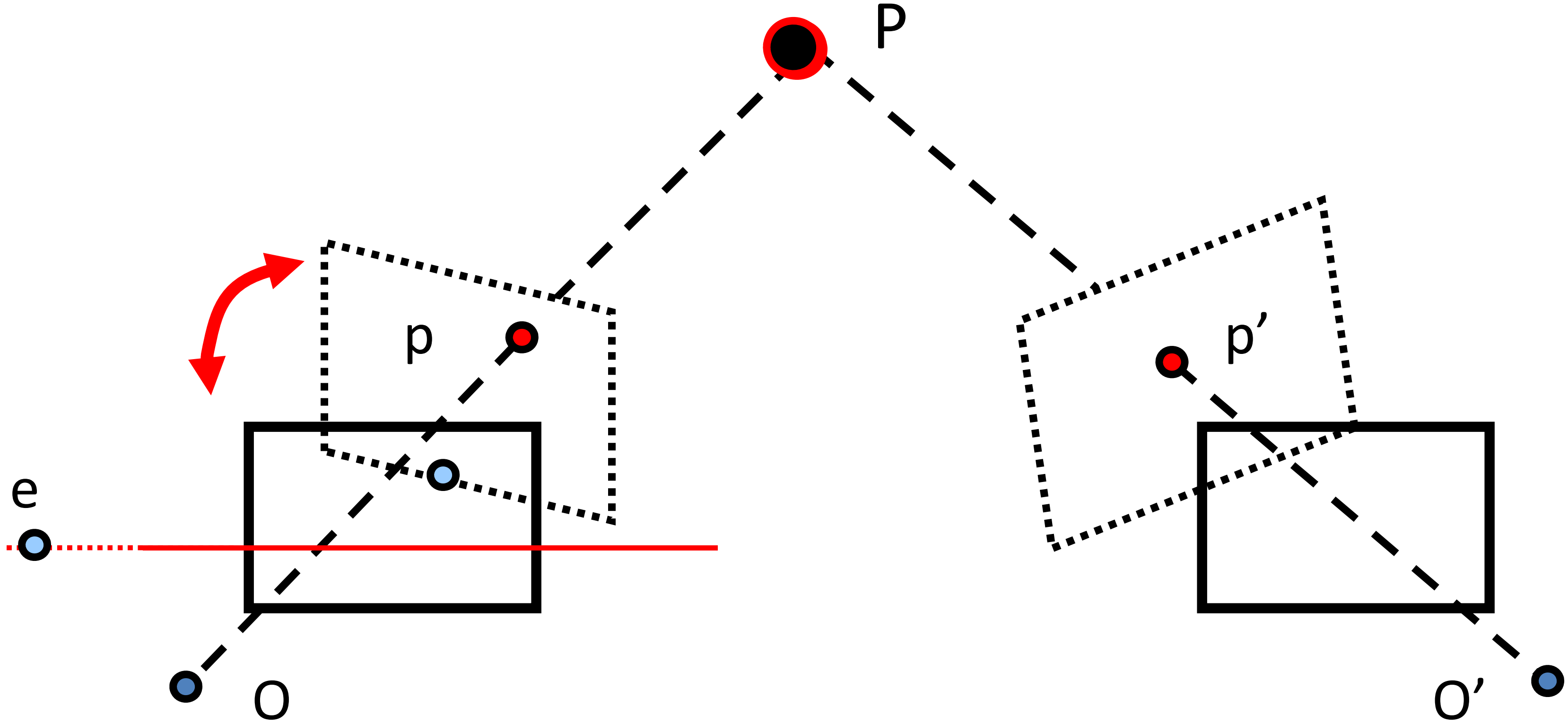


1. Map e to the x-axis at location $[1,0,1]^T$ (normalization)

$$e = [e_1 \quad e_2 \quad 1]^T \rightarrow [1 \quad 0 \quad 1]^T$$

$$\boxed{H_1 = R_H T_H}$$

Making image planes parallel



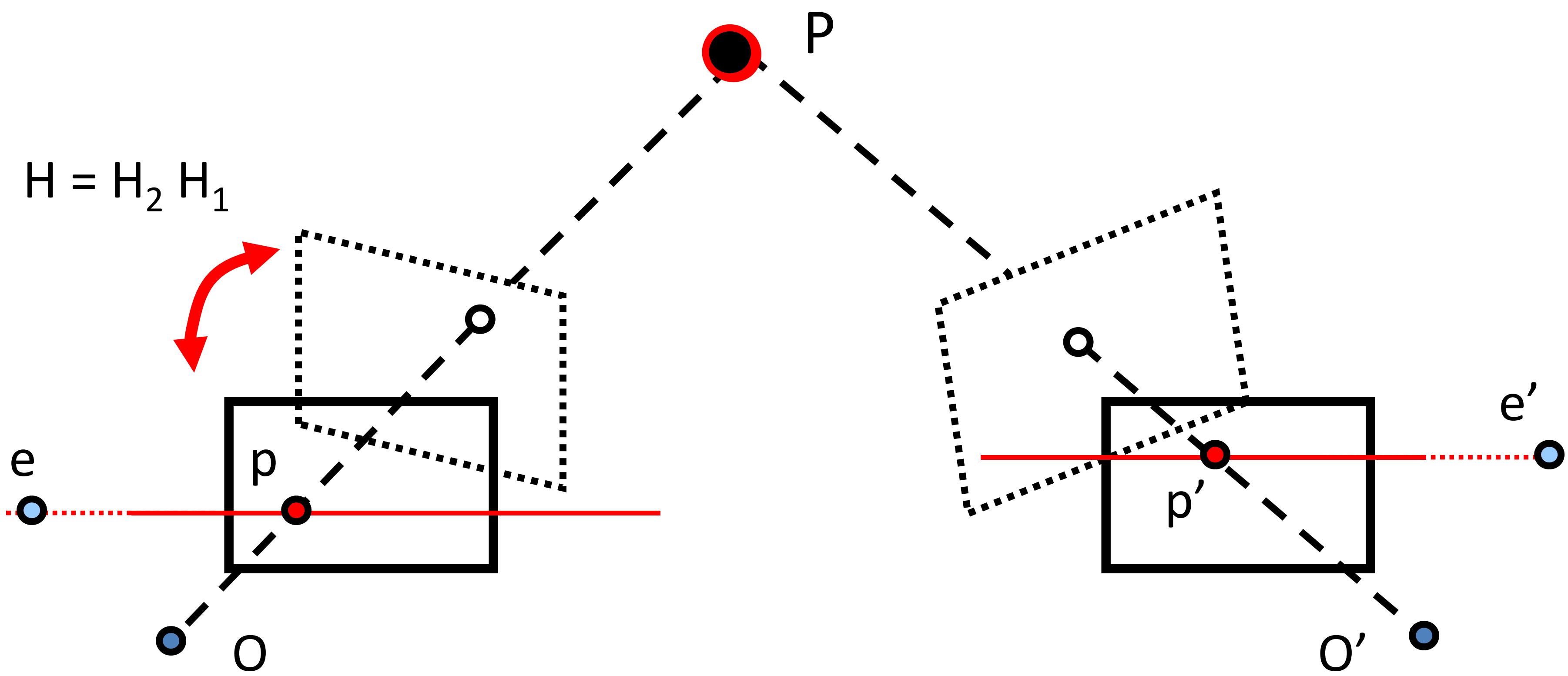
2. Send epipole to infinity:

$$e = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \rightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

$$\boxed{H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}$$

Minimizes the distortion in a neighborhood (approximates id. mapping)

Making image planes parallel

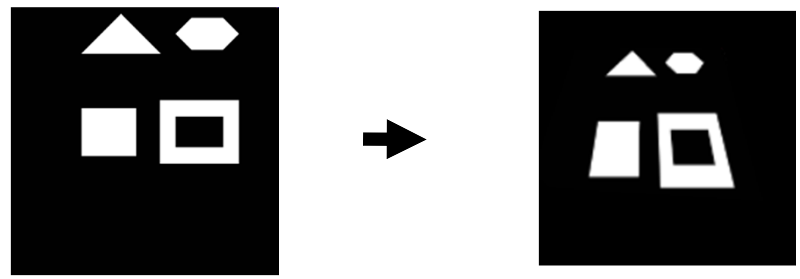


3. Define: $H = H_2 H_1$

4. Align epipolar lines

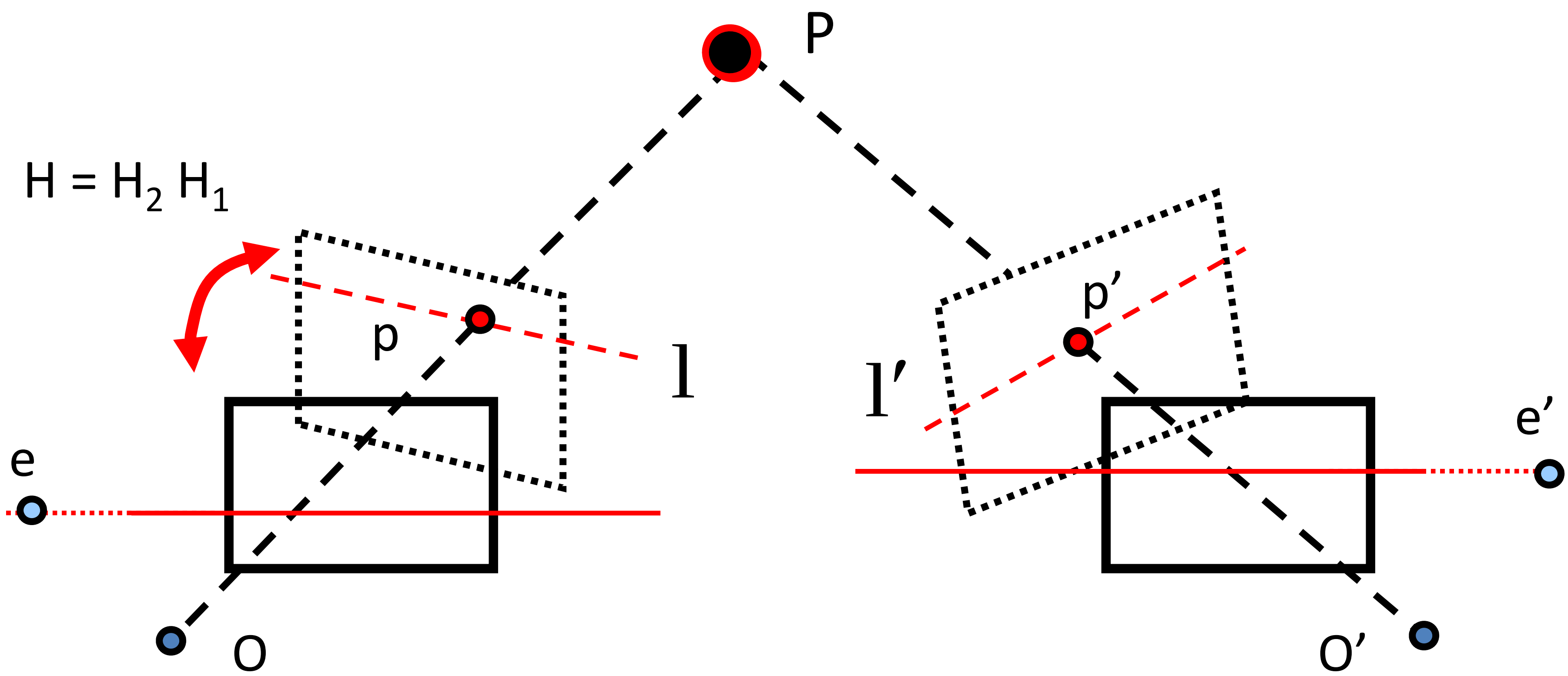
Projective transformation of a line (in 2D)

$$H = \begin{bmatrix} A & t \\ v & b \end{bmatrix}$$



$$l \rightarrow H^{-T}\, l$$

Making image planes parallel



3. Define: $H = H_2 H_1$

4. Align epipolar lines

[HZ] Chapters: 11 (sec. 11.12)

$$\overline{H'}^{-T} l' = \overline{H}^{-T} \, l$$

These are called matched pair of transformation