Lecture 9: Epipolar Geometry

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What we will learn today?

• Why is stereo useful?
• Epipolar constraints
• Essential and fundamental matrix
• Estimating F (Problem Set 2 (Q2))
• Rectification

Reading:
[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10
Recovering structure from a single view

Why is it so difficult?

Intrinsic ambiguity of the mapping from 3D to image (2D)
Recovering structure from a single view

Intrinsic ambiguity of the mapping from 3D to image (2D)
Two eyes help!
Two eyes help!

This is called triangulation
Triangulation

- Find $X$ that minimizes $d^2(x_1, P_1X) + d^2(x_2, P_2X)$
Stereo-view geometry

- **Correspondence:** Given a point in one image, how can I find the corresponding point $x'$ in another one?

- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.

- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.
Stereo-view geometry

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Next lecture (#10)
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Reading:
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Epipolar geometry

- Epipolar Plane
- Baseline
- Epipolar Lines

Epipoles $e_1$, $e_2$

= intersections of baseline with image planes
= projections of the other camera center
= vanishing points of camera motion direction
Example: Converging image planes
Example: Parallel image planes

- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to x axis
Example: Parallel image planes

\begin{align*}
e & \text{ at} \\
\text{infinity} & \\

\end{align*}
Example: Forward translation

- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)
- Two views of the same object
- Suppose I know the camera positions and camera matrices
- Given a point on left image, how can I find the corresponding point on right image?
Epipolar Constraint

- Potential matches for $p$ have to lie on the corresponding epipolar line $l'$. 
- Potential matches for $p'$ have to lie on the corresponding epipolar line $l$. 

Epipolar Constraint

\[ p \rightarrow MP = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \]

\[ p \rightarrow M'P = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} \]

\[ M = K[I \ 0] \]

\[ M' = K[R \ T] \]
Epipolar Constraint

\[ M = K[I \ 0] \]

\[ M = [I \ 0] \]

\[ K_1 \text{ and } K_2 \text{ are known (calibrated cameras)} \]

\[ M' = K[R \ T] \]

\[ M' = [R \ T] \]
Epipolar Constraint

\[ T \times (R \ p') \]
Perpendicular to epipolar plane

\[ p^T \cdot \left[ T \times (R \ p') \right] = 0 \]
Cross product as matrix multiplication

\[ \mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a} \times] \mathbf{b} \]

“skew symmetric matrix”
Triangulation

\[ p^T \cdot \left[ T \times (R \cdot p') \right] = 0 \rightarrow p^T \cdot \left[ T_x \right] \cdot R \cdot p' = 0 \]

(Longuet-Higgins, 1981)  
\( E = \text{essential matrix} \)

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Triangulation

- $E \ p_2$ is the epipolar line associated with $p_2$ ($l_1 = E \ p_2$)
- $E^T p_1$ is the epipolar line associated with $p_1$ ($l_2 = E^T p_1$)
- $E$ is singular (rank two)
- $E \ e_2 = 0$ and $E^T e_1 = 0$
- $E$ is $3 \times 3$ matrix; 5 DOF
Triangulation

\[ P \rightarrow M \rightarrow p = \begin{bmatrix} u \\ v \end{bmatrix} \]

\[ M = K \begin{bmatrix} I & 0 \end{bmatrix} \]

unknown
Triangulation

\[ p \rightarrow K^{-1} p \]

\[
p^T \cdot [T_x] \cdot R \ p' = 0 \rightarrow (K^{-1} p)^T \cdot [T_x] \cdot R \ K'^{-1} \ p' = 0
\]

\[
p^T \begin{bmatrix} K^{-T} \cdot [T_x] \cdot R \ K'^{-1} \end{bmatrix} \ p' = 0 \rightarrow p^T \begin{bmatrix} F \end{bmatrix} \ p' = 0
\]
Triangulation

\[ p^T F p' = 0 \]

**F = Fundamental Matrix**

(Faugeras and Luong, 1992)
Triangulation

- $F p_2$ is the epipolar line associated with $p_2$ ($l_1 = F p_2$)
- $F^T p_1$ is the epipolar line associated with $p_1$ ($l_2 = F^T p_1$)
- $F$ is singular (rank two)
- $Fe_2 = 0$ and $F^T e_1 = 0$
- $F$ is 3x3 matrix; 7 DOF
Why is F useful?

- Suppose F is known
- No additional information about the scene and camera is given
- Given a point on left image, how can I find the corresponding point on right image?

\[ l' = F^T p \]
Why is $F$ useful?

- $F$ captures information about the epipolar geometry of 2 views + camera parameters

- **MORE IMPORTANTLY:** $F$ gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)

- Powerful tool in:
  - 3D reconstruction
  - Multi-view object/scene matching
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Reading:
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- [FP] Chapters: 10
The Eight-Point Algorithm

(Longuet-Higgins, 1981)
(Hartley, 1995)

\[ P \rightarrow p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad \text{P} \rightarrow p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} \quad p^T F p' = 0 \]
Estimating $F$

$$p^\top F p' = 0 \quad \Rightarrow \quad \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

$$\quad \Rightarrow \quad (uu', uv', u, vu', vv', v, u', v', 1) \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

Let’s take 8 corresponding points
Estimating F
Estimating $\mathbf{F}$

$$\begin{pmatrix}
  u_1 u_1' & u_1 v_1' & u_1 & v_1 u_1' & v_1 v_1' & v_1 & u_1' & v_1 v_1' \v 1 \\
  u_2 u_2' & u_2 v_2' & u_2 & v_2 u_2' & v_2 v_2' & v_2 & u_2' & v_2 v_2' \v 2 \\
  u_3 u_3' & u_3 v_3' & u_3 & v_3 u_3' & v_3 v_3' & v_3 & u_3' & v_3 v_3' \v 3 \\
  u_4 u_4' & u_4 v_4' & u_4 & v_4 u_4' & v_4 v_4' & v_4 & u_4' & v_4 v_4' \v 4 \\
  u_5 u_5' & u_5 v_5' & u_5 & v_5 u_5' & v_5 v_5' & v_5 & u_5' & v_5 v_5' \v 5 \\
  u_6 u_6' & u_6 v_6' & u_6 & v_6 u_6' & v_6 v_6' & v_6 & u_6' & v_6 v_6' \v 6 \\
  u_7 u_7' & u_7 v_7' & u_7 & v_7 u_7' & v_7 v_7' & v_7 & u_7' & v_7 v_7' \v 7 \\
  u_8 u_8' & u_8 v_8' & u_8 & v_8 u_8' & v_8 v_8' & v_8 & u_8' & v_8 v_8' \v 8 \\
\end{pmatrix} \begin{pmatrix}
  \mathbf{F}_{11} \\
  \mathbf{F}_{12} \\
  \mathbf{F}_{13} \\
  \mathbf{F}_{21} \\
  \mathbf{F}_{22} \\
  \mathbf{F}_{23} \\
  \mathbf{F}_{31} \\
  \mathbf{F}_{32} \\
  \mathbf{F}_{33} \\
\end{pmatrix} = \mathbf{0}$$

- Homogeneous system $\mathbf{W} \mathbf{f} = 0$
- Rank $8$, A non-zero solution exists (unique)
- If $N>8$, Lsq. solution by SVD! $\mathbf{F}$, $\|\mathbf{f}\| = 1$
Estimating $F$

$$p^T \hat{F} p' = 0$$

The estimated $F$ may have full rank ($\det(F) \neq 0$) ($F$ should have rank=2 instead)

Find $F$ that minimizes

$$\|F - \hat{F}\| = 0$$

Frobenius norm (*)

Subject to $\det(F)=0$

SVD (again!) can be used to solve this problem

(*) Sqrt root of the sum of squares of all entries
Example

Data courtesy of R. Mohr and B. Boufama.
Example

Mean errors:
10.0pixel
9.1pixel
Normalization

Is the accuracy in estimating $F$ function of the ref. system in the image plane?

E.g. under similarity transformation ($T = \text{scale} + \text{translation}$):

$$q_i = T_i \ p_i \quad q_i' = T_i' \ p_i'$$

Does the accuracy in estimating $F$ change if a transformation $T$ is applied?
Normalization

• The accuracy in estimating F does change if a transformation T is applied.

• There exists a T for which accuracy is maximized.

Why?
Normalization

- SVD enforces $\text{Rank}(W) = 8$
- Recall the structure of $W$: Highly un-balance (not well conditioned)
- Values of $W$ must have similar magnitude

More details HZ pag 108
Normalization

**IDEA:** Transform image coordinate system \((T = \text{translation} + \text{scaling})\) such that:

- Origin = centroid of image points
- Mean square distance of the data points from origin is 2 pixels

\[
q_i = T_i \, p_i \quad q'_i = T'_i \, p'_i \quad \text{(normalization)}
\]
The Normalized Eight-Point Algorithm

0. Compute $T_i$ and $T_i'$

1. Normalize coordinates:

$$q_i = T_i \ p_i \quad \quad q'_i = T'_i \ p'_i$$

2. Use the eight-point algorithm to compute $F'_q$ from the points $q_i$ and $q'_i$.

3. Enforce the rank-2 constraint.

$$\rightarrow \ F_q \quad \left\{ \begin{align*}
q^T F_q \ q' &= 0 \\
\det(F_q) &= 0
\end{align*} \right. $$

4. De-normalize $F_q$:

$$F = T'^T F_q T$$
Example

Without transformation

Mean errors: 10.0 pixel
9.1 pixel

With transformation

Mean errors: 1.0 pixel
0.9 pixel
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Reading:
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[FP] Chapters: 10
Rectification

- Make two camera images “parallel”
- Correspondence problem becomes easier
Rectification

- Parallel epipolar lines
- Epipoles at infinity
- \( v = v' \)

Let’s see why....
Recification

\[ K_1 = K_2 = \text{known} \]
\[ x \text{ parallel to } O_1O_2 \]

\[ E = [t_x]R \]
Cross product as matrix multiplication

\[ \mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_x] \mathbf{b} \]
Rectification

\[ E = [t_x]R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -T & 0 \\ 0 & T & 0 & 0 \end{bmatrix} \quad \rightarrow v = v' ? \]

\( K_1 = K_2 = \text{known} \)

\( x \) parallel to \( O_1O_2 \)
Rectification

\[
(p')^T E p' = 0
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -T \\
0 & T & 0
\end{bmatrix}
\begin{bmatrix}
u' \\
v' \\
1
\end{bmatrix} = 0
\]

\[
(u \quad v \quad 1)
\begin{bmatrix}
0 \\
-T \\
T
\end{bmatrix}
\begin{bmatrix}
0 \\
Tv' \\
Tv'
\end{bmatrix} = 0
\]

\[Tv = Tv' \rightarrow v = v'\]
**GOAL of rectification**: Estimate a perspective transformation $H$ that makes images parallel

- **Impose $v' = v$**

- This leaves degrees of freedom for determining $H$
- If not appropriate $H$ is chosen, severe projective distortions on image take place
- We impose a number of restriction while computing $H$

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[HZ] Chapters: 11 (sec. 11.12)
Rectification

 Courtesy figure S. Lazebnik
Application: view morphing

Application: view morphing

If rectification is not applied, the morphing procedure does not generate geometrically correct interpolations.
Application: view morphing
Application: view morphing
Application: view morphing
Application: view morphing
The Fundamental Matrix Song

http://danielwedge.com/fmatrix/
What we have learned today?

- Why is stereo useful?
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Reading:
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Supplementary materials
Making image planes parallel

\[ P \rightarrow K[I \ 0] P \]

0. Compute epipoles

\[ e = KR^T T = [e_1 \ e_2 \ 1]^T \quad e' = K'T \]

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Making image planes parallel

1. Map $e$ to the x-axis at location $[1,0,1]^T$ (normalization)

$$e = [e_1 \ e_2 \ 1]^T \rightarrow \begin{bmatrix} e_1 & e_2 & 1 \end{bmatrix}^T$$

$$H_1 = R_H T_H$$
Making image planes parallel

2. Send epipole to infinity:

\[ e = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \rightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \]

Minimizes the distortion in a neighborhood (approximates id. mapping)
Making image planes parallel

3. Define: $H = H_2 H_1$

4. Align epipolar lines
Projective transformation of a line (in 2D)

\[ H = \begin{bmatrix} A & t \\ v & b \end{bmatrix} \]

\[ l \rightarrow H^{-T} l \]
Making image planes parallel

3. Define: $H = H_2 H_1$

4. Align epipolar lines

$\overline{H'} - T \overline{l'} = \overline{H} - T \overline{l}$

These are called matched pair of transformation

[HZ] Chapters: 11 (sec. 11.12)