What we will learn today?

• Why is stereo useful?
• Epipolar constraints
• Essential and fundamental matrix
• Estimating F (Problem Set 2 (Q2))
• Rectification

Reading:
[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10
Recovering structure from a single view

Why is it so difficult?

Intrinsic ambiguity of the mapping from 3D to image (2D)
Recovering structure from a single view

Intrinsic ambiguity of the mapping from 3D to image (2D)
Two eyes help!
Two eyes help!

This is called triangulation
Triangulation

• Find $X$ that minimizes $d^2(x_1, P_1X) + d^2(x_2, P_2X)$
Stereo-view geometry

- **Correspondence**: Given a point in one image, how can I find the corresponding point $x'$ in another one?

- **Camera geometry**: Given corresponding points in two images, find camera matrices, position and pose.

- **Scene geometry**: Find coordinates of 3D point from its projection into 2 or multiple images.
Stereo-view geometry

Next lecture (#10)

• **Correspondence:** Given a point in one image, how can I find the corresponding point $x'$ in another one?

• **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.

• **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.
What we will learn today?

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F
- Rectification

Reading:
[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10
Epipolar geometry

- Epipolar Plane
- Baseline
- Epipolar Lines

- Epipoles \( e_1, e_2 \)
  - intersections of baseline with image planes
  - projections of the other camera center
  - vanishing points of camera motion direction
Example: Converging image planes
Example: Parallel image planes

- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to x axis
Example: Parallel image planes
Example: Forward translation

- The epipoles have the same position in both images.
- Epipole called FOE (focus of expansion).
Epipolar Constraint

- Two views of the same object
- Suppose I know the camera positions and camera matrices
- Given a point on left image, how can I find the corresponding point on right image?
Epipolar Constraint

- Potential matches for $p$ have to lie on the corresponding epipolar line $l'$. 
- Potential matches for $p'$ have to lie on the corresponding epipolar line $l$. 
Epipolar Constraint

\[ p \rightarrow M \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \]

\[ p' \rightarrow M' \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} \]

\[ M = K \begin{bmatrix} I & 0 \end{bmatrix} \]

\[ M' = K \begin{bmatrix} R & T \end{bmatrix} \]
Epipolar Constraint

\[
M = K[I \ 0] \\
M' = K[R \ T]
\]

\[
M = [I \ 0] \\
M' = [R \ T]
\]

K₁ and K₂ are known (calibrated cameras)
Epipolar Constraint

\[ T \times (R \ p') \]

Perpendicular to epipolar plane

\[ p^T \cdot [T \times (R \ p')] = 0 \]
Cross product as matrix multiplication

\[ \mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a} \times] \mathbf{b} \]

“skew symmetric matrix”
p^T \cdot \left[ T \times (R \, p') \right] = 0 \quad \rightarrow \quad p^T \cdot \left[ T_\times \right] \cdot R \, p' = 0

(Longuet-Higgins, 1981) \quad E = \text{essential matrix}
Triangulation

- $E \mathbf{p}_2$ is the epipolar line associated with $\mathbf{p}_2$ ($l_1 = E \mathbf{p}_2$)
- $E^T \mathbf{p}_1$ is the epipolar line associated with $\mathbf{p}_1$ ($l_2 = E^T \mathbf{p}_1$)
- $E$ is singular (rank two)
- $E \mathbf{e}_2 = 0$ and $E^T \mathbf{e}_1 = 0$
- $E$ is a 3x3 matrix; 5 DOF
Triangulation

\[ P \rightarrow M \quad P \rightarrow p = \begin{bmatrix} u \\ v \end{bmatrix} \quad M = \begin{bmatrix} K & I & 0 \end{bmatrix} \]

unknown
Triangulation

\[ p \rightarrow K^{-1} p \]

\[ p^T \cdot [T_x] \cdot R \ p' = 0 \rightarrow (K^{-1} p)^T \cdot [T_x] \cdot R \ K'^{-1} \ p' = 0 \]

\[ p^T \begin{bmatrix} K^{-T} \cdot [T_x] \cdot R \ K'^{-1} \end{bmatrix} \ p' = 0 \rightarrow p^T \begin{bmatrix} F \end{bmatrix} \ p' = 0 \]
Triangulation

\[ p^T F p' = 0 \]

\[ F = \text{Fundamental Matrix} \]

(Faugeras and Luong, 1992)
Triangulation

- \( F p_2 \) is the epipolar line associated with \( p_2 \) (\( l_1 = F p_2 \))
- \( F^T p_1 \) is the epipolar line associated with \( p_1 \) (\( l_2 = F^T p_1 \))
- \( F \) is singular (rank two)
- \( F e_2 = 0 \) and \( F^T e_1 = 0 \)
- \( F \) is a 3x3 matrix; 7 DOF
Why is $F$ useful?

- Suppose $F$ is known
- No additional information about the scene and camera is given
- Given a point on left image, how can I find the corresponding point on right image?

\[ l' = F^T p \]
Why is F useful?

- F captures information about the epipolar geometry of 2 views + camera parameters

- **MORE IMPORTANTLY:** F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)

- Powerful tool in:
  - 3D reconstruction
  - Multi-view object/scene matching
What we will learn today?

• Why is stereo useful?
• Epipolar constraints
• Essential and fundamental matrix
• Estimating $F$ (Problem Set 2 (Q2))
• Rectification

Reading:
[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10
Estimating $F$

The Eight-Point Algorithm

(Longuet-Higgins, 1981)
(Hartley, 1995)

$$p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

$$p^T F p' = 0$$
Estimating \( F \)

\[
p^T F p' = 0
\]

Let’s take 8 corresponding points
Estimating F
Estimating $F$:

\[
\begin{pmatrix}
 u_1 u'_1 & u_1 v'_1 & u_1 & v_1 u'_1 & v_1 v'_1 & v_1 & u'_1 & v'_1 & 1 \\
 u_2 u'_2 & u_2 v'_2 & u_2 & v_2 u'_2 & v_2 v'_2 & v_2 & u'_2 & v'_2 & 1 \\
 u_3 u'_3 & u_3 v'_3 & u_3 & v_3 u'_3 & v_3 v'_3 & v_3 & u'_3 & v'_3 & 1 \\
 u_4 u'_4 & u_4 v'_4 & u_4 & v_4 u'_4 & v_4 v'_4 & v_4 & u'_4 & v'_4 & 1 \\
 u_5 u'_5 & u_5 v'_5 & u_5 & v_5 u'_5 & v_5 v'_5 & v_5 & u'_5 & v'_5 & 1 \\
 u_6 u'_6 & u_6 v'_6 & u_6 & v_6 u'_6 & v_6 v'_6 & v_6 & u'_6 & v'_6 & 1 \\
 u_7 u'_7 & u_7 v'_7 & u_7 & v_7 u'_7 & v_7 v'_7 & v_7 & u'_7 & v'_7 & 1 \\
 u_8 u'_8 & u_8 v'_8 & u_8 & v_8 u'_8 & v_8 v'_8 & v_8 & u'_8 & v'_8 & 1 \\
\end{pmatrix}
\begin{pmatrix}
 F_{11} \\
 F_{12} \\
 F_{13} \\
 F_{21} \\
 F_{22} \\
 F_{23} \\
 F_{31} \\
 F_{32} \\
 F_{33} \\
\end{pmatrix}
= 0
\]

- Homogeneous system \( W f = 0 \)
- Rank 8 \( \rightarrow \) A non-zero solution exists (unique)
- If \( N > 8 \) \( \rightarrow \) Lsq. solution by SVD! \( \rightarrow \) \( \hat{F} \) \( \|f\| = 1 \)
Estimating $F$

$$p^T \hat{F} p' = 0$$

The estimated $\hat{F}$ may have full rank ($\det(F) \neq 0$) (F should have rank=2 instead)

Find $F$ that minimizes

$$\left\| F - \hat{F} \right\| = 0$$

Frobenius norm (*)

Subject to $\det(F)=0$

SVD (again!) can be used to solve this problem

(*) Sqrt root of the sum pf squares of all entries
Example

Data courtesy of R. Mohr and B. Boufama.
Example

Mean errors:
10.0 pixel
9.1 pixel
Normalization

Is the accuracy in estimating $F$ function of the ref. system in the image plane?

E.g. under similarity transformation ($T = \text{scale} + \text{translation}$):

$$q_i = T_i \ p_i \quad q'_i = T'_i \ p'_i$$

Does the accuracy in estimating $F$ change if a transformation $T$ is applied?
Normalization

• The accuracy in estimating F does change if a transformation T is applied

• There exists a T for which accuracy is maximized

Why?
Normalization

\[ W \mathbf{f} = 0, \quad \| \mathbf{f} \| = 1 \]

- SVD enforces \( \text{Rank}(W) = 8 \)
- Recall the structure of \( W \): Highly un-balance (not well conditioned)
- Values of \( W \) must have similar magnitude

More details HZ pag 108
**Normalization**

**IDEA:** Transform image coordinate system ($T = \text{translation} + \text{scaling}$) such that:

- Origin = centroid of image points
- Mean square distance of the data points from origin is 2 pixels

$$q_i = T_i \ p_i$$
$$q'_i = T'_i \ p'_i$$ (normalization)
The Normalized Eight-Point Algorithm

0. Compute $T_i$ and $T_i'$

1. Normalize coordinates:
   
   $$q_i = T_i \ p_i \quad q'_i = T'_i \ p'_i$$

2. Use the eight-point algorithm to compute $F'_q$ from the points $q_i$ and $q'_i$.

3. Enforce the rank-2 constraint. 

   $$F \rightarrow F'_q \quad \begin{cases} 
   q^T F_q \ q' = 0 \\
   \det( F_q ) = 0 
   \end{cases}$$

4. De-normalize $F_q$:

   $$F = T'^T F_q \ T$$
Example

Mean errors:
Without transformation: 10.0 pixel, 9.1 pixel
With transformation: 1.0 pixel, 0.9 pixel
What we will learn today?

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating $F$
- Rectification

Reading:

[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10
Rectification

• Make two camera images “parallel”
  • Correspondence problem becomes easier
Rectification

- Parallel epipolar lines
- Epipoles at infinity
- $v = v'$

Let's see why....
Rectification

\[ K_1 = K_2 = \text{known} \]

\[ x \text{ parallel to } O_1O_2 \]

\[ E = [t_x]R \]
Cross product as matrix multiplication

\[
\mathbf{a} \times \mathbf{b} = \begin{bmatrix}
0 & -a_z & a_y \\
a_z & 0 & -a_x \\
-a_y & a_x & 0
\end{bmatrix}
\begin{bmatrix}
b_x \\
b_y \\
b_z
\end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}
\]
Rectification

\[ K_1 = K_2 = \text{known} \]
\[ x \text{ parallel to } O_1O_2 \]

\[ E = [t_x]R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \]

\[ \rightarrow v = v' ? \]
Rectification

\[ p^T E p' = 0 \]

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -T \\
0 & T & 0 \\
\end{pmatrix}
\begin{pmatrix}
u' \\
v' \\
1 \\
\end{pmatrix} = 0
\]

\[
(u \ v \ 1)
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -T \\
0 & T & 0 \\
\end{pmatrix}
\begin{pmatrix}
u' \\
v' \\
1 \\
\end{pmatrix} = 0
\]

\[
(u \ v \ 1)
\begin{pmatrix}
0 \\
-\cancel{T} \\
\cancel{T}v' \\
\end{pmatrix}
= 0
\]

\[ T v = T v' \]

\[ \rightarrow v = v' \]
**GOAL of rectification** : Estimate a perspective transformation $H$ that makes images parallel

- Impose $v' = v$

- This leaves degrees of freedom for determining $H$
- If not appropriate $H$ is chosen, severe projective distortions on image take place
- We impose a number of restriction while computing $H$

[HZ] Chapters: 11 (sec. 11.12)
Rectification

Courtesy figure S. Lazebnik
Application: view morphing

Application: view morphing

If rectification is not applied, the morphing procedure does not generate geometrically correct interpolations.
Application: view morphing
Application: view morphing
Application: view morphing
Application: view morphing
The Fundamental Matrix Song

http://danielwedge.com/fmatrix/
What we have learned today?

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating $F$ ([Problem Set 2 (Q2)])
- Rectification

Reading:

[HZ] Chapters: 4, 9, 11
[FP] Chapters: 10
Supplementary materials
Making image planes parallel

\[ \mathbf{P} \rightarrow \mathbf{K} \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{P} \]

\[ \mathbf{P} \rightarrow \mathbf{K}' \begin{bmatrix} \mathbf{R} & \mathbf{T} \end{bmatrix} \mathbf{P} \]

0. Compute epipoles

\[ \mathbf{e} = \mathbf{K} \mathbf{R}^T \mathbf{T} = [e_1, e_2, 1]^T \]

\[ \mathbf{e}' = \mathbf{K}' \mathbf{T} \]
Making image planes parallel

1. Map $e$ to the x-axis at location $[1,0,1]^T$ (normalization)

$$e = [e_1, e_2, 1]^T \rightarrow \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$$

$$H_1 = R_H T_H$$
Making image planes parallel

2. Send epipole to infinity:

\[ e = [1 \ 0 \ 1]^T \rightarrow [1 \ 0 \ 0]^T \]

Minimizes the distortion in a neighborhood (approximates id. mapping)

\[ H_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix} \]
3. Define: $H = H_2 H_1$

4. Align epipolar lines
Projective transformation of a line (in 2D)

$$H = \begin{bmatrix} A & t \\ v & b \end{bmatrix} \quad \rightarrow \quad \begin{array}{c}
\end{array}
$$

$$l \rightarrow H^{-T} l$$
Making image planes parallel

3. Define: \( H = H_2 H_1 \)

4. Align epipolar lines

\[ \overline{H}'^{-T} l' = \overline{H}^{-T} l \]

These are called matched pair of transformation

[HZ] Chapters: 11 (sec. 11.12)