Lecture 2
Camera Models

Professor Silvio Savarese

Computational Vision and Geometry Lab

Announcements

Prerequisites: any questions?

This course requires knowledge of linear algebra, probability, statistics, machine learning and computer vision, as well as decent programming skills. Though not an absolute requirement, it is encouraged and preferred that you have at least taken either CS221 or CS229 or CS131A or have equivalent knowledge.

Topics such as **linear filters, feature detectors and descriptors, low level segmentation, tracking, optical flow, clustering and PCA/LDA techniques** for recognition won't be covered in CS231

We will provide links to background material related to CS131A (or discuss during TA sessions) so students can refresh or study those topics if needed

We will leverage concepts from machine learning (CS229) (e.g., **SVM, basic Bayesian inference, clustering**, etc…) which we won't cover in this class either. Again, we will supply links to related material for background reading.

Announcements

Next TA session: Fridays from 2:15-3:05pm

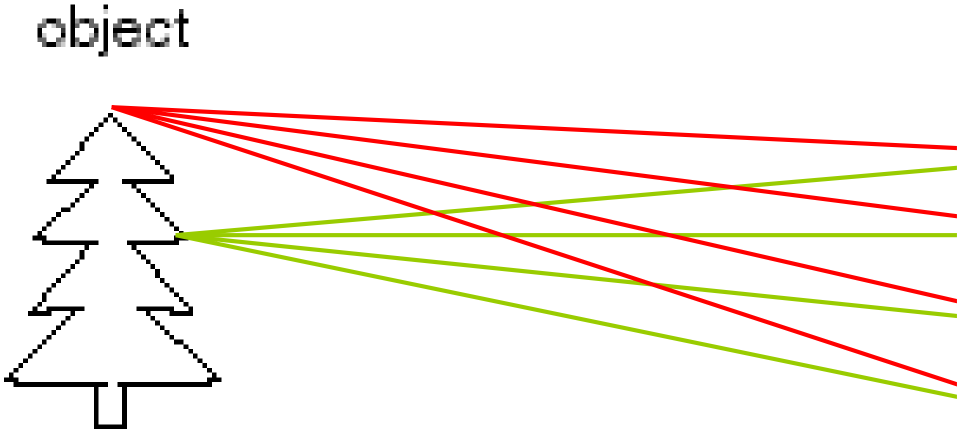
Lecture 2
Camera Models

- Pinhole cameras
- Cameras & lenses
- The geometry of pinhole cameras
- Other camera models

Reading: [FP] Chapter 1 “Cameras”
[FP] Chapter 2 “Geometric Camera Models”
[HZ] Chapter 6 “Camera Models”

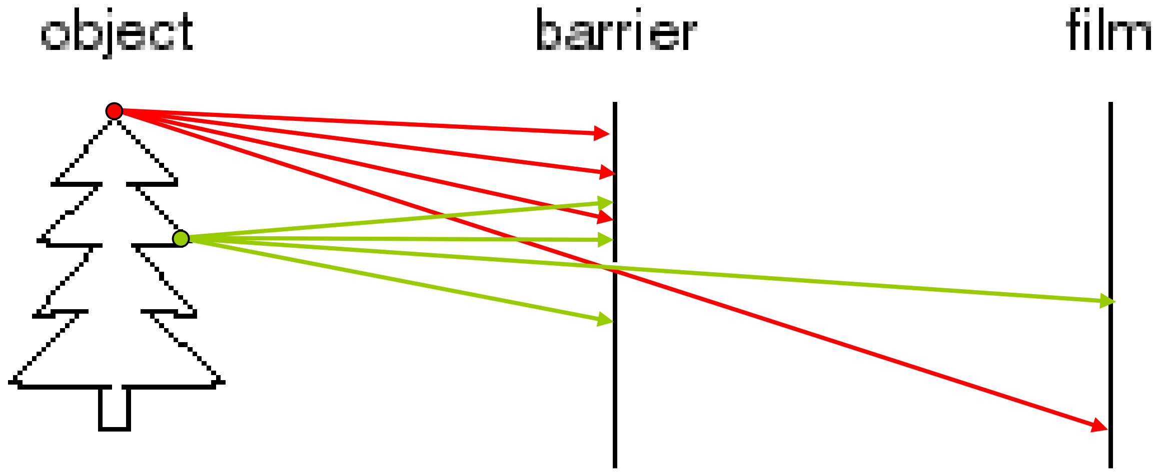
Some slides in this lecture are courtesy to Profs. J. Ponce, S. Seitz, F-F Li

How do we see the world?



- Let’s design a camera
  - Idea 1: put a piece of film in front of an object
  - Do we get a reasonable image?

Pinhole camera



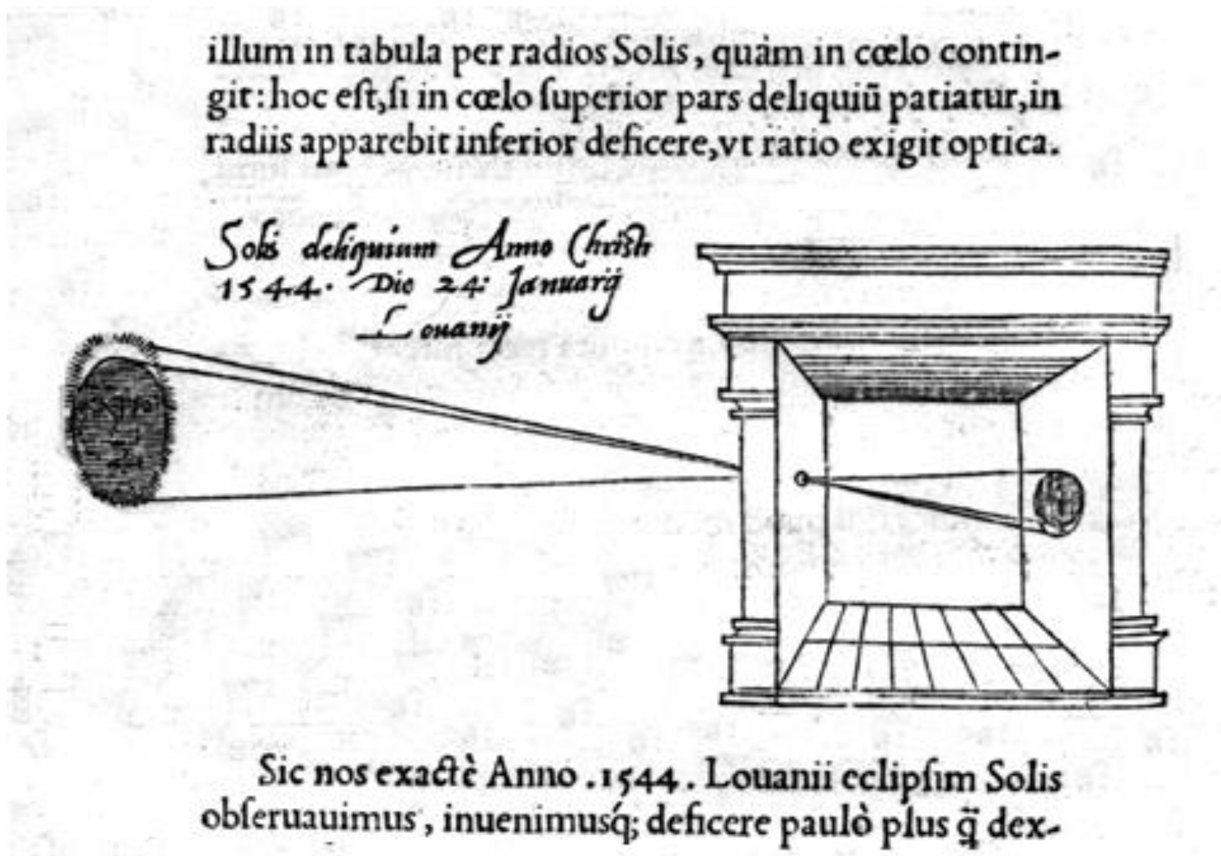
- Add a barrier to block off most of the rays
  - This reduces blurring
  - The opening known as the **aperture**

Some history…

Milestones:

• Leonardo da Vinci (1452-1519):

first record of camera *obscura*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Sic nos exactè Anno .1544. Louanii eclipſim Solis
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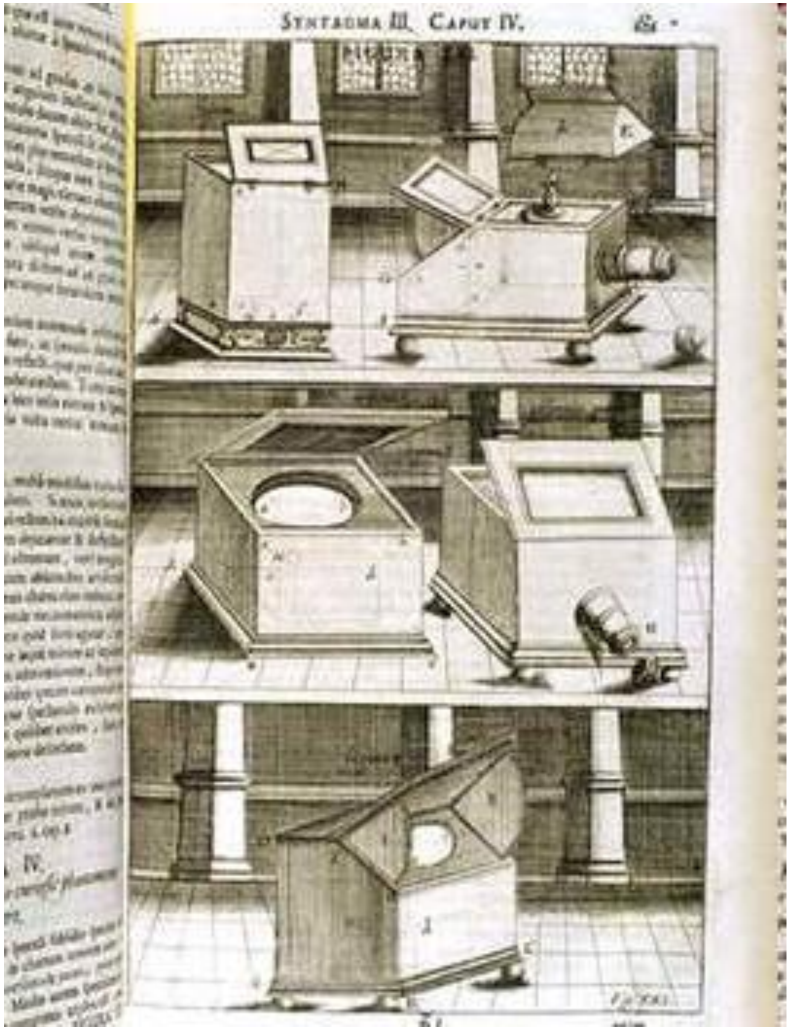
Some history…

Milestones:

- Leonardo da Vinci (1452-1519): first record of camera *obscura*
- Johann Zahn (1685): first portable camera



Some history…

Milestones:

• Leonardo da Vinci (1452-1519): first record of camera *obscura*

• Johann Zahn (1685): first portable camera

• Joseph Nicephore Niepce (1822): first photo - birth of photography

Photography (Niepce, “La Table Servie,” 1822)

Some history…

Milestones:

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• Daguerréotypes (1839)

• Photographic Film (Eastman, 1889)

• Cinema (Lumière Brothers, 1895)

• Color Photography (Lumière Brothers, 1908)

Photography (Niepce, “La Table Servie,” 1822)

Let’s also not forget…

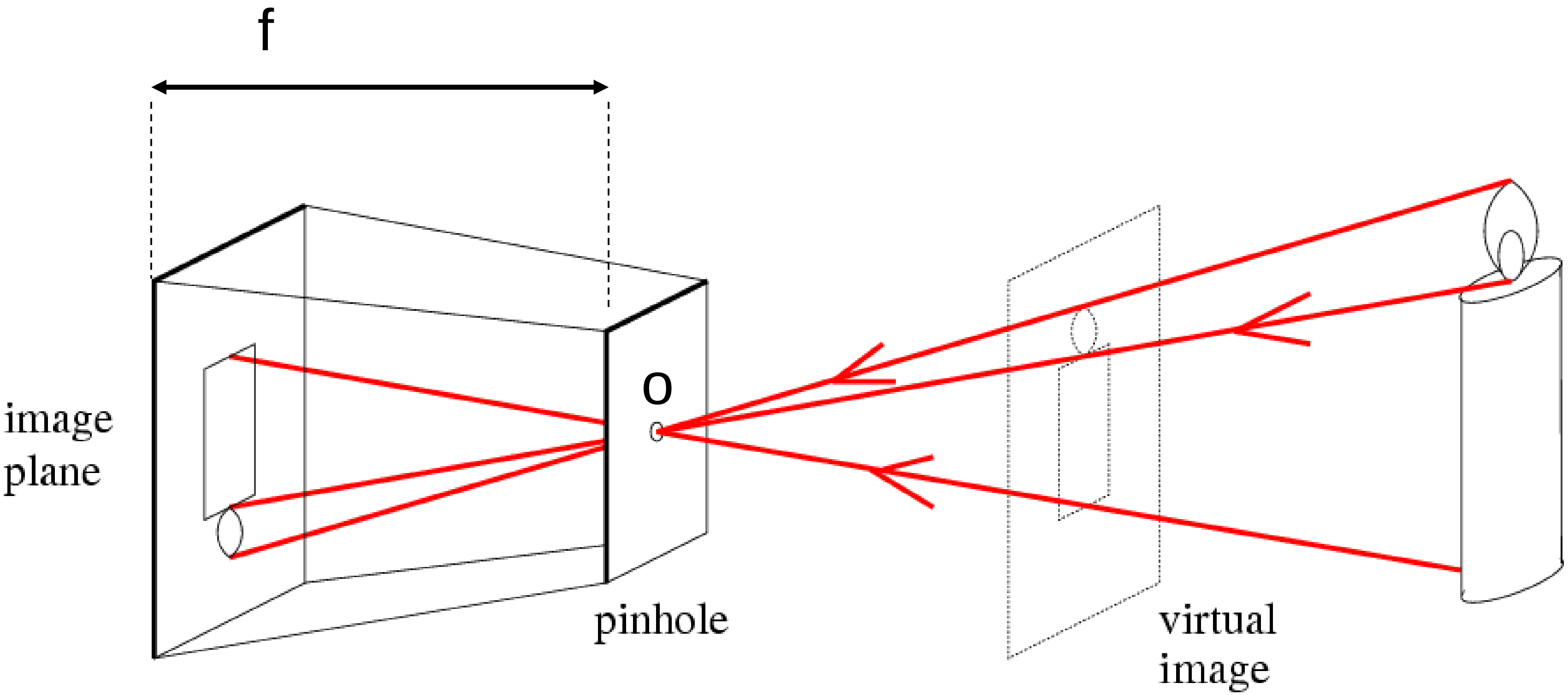
Motzu
(468-376 BC)

Oldest existent book on geometry in China

Aristotle
(384-322 BC)
Also: Plato, Euclid

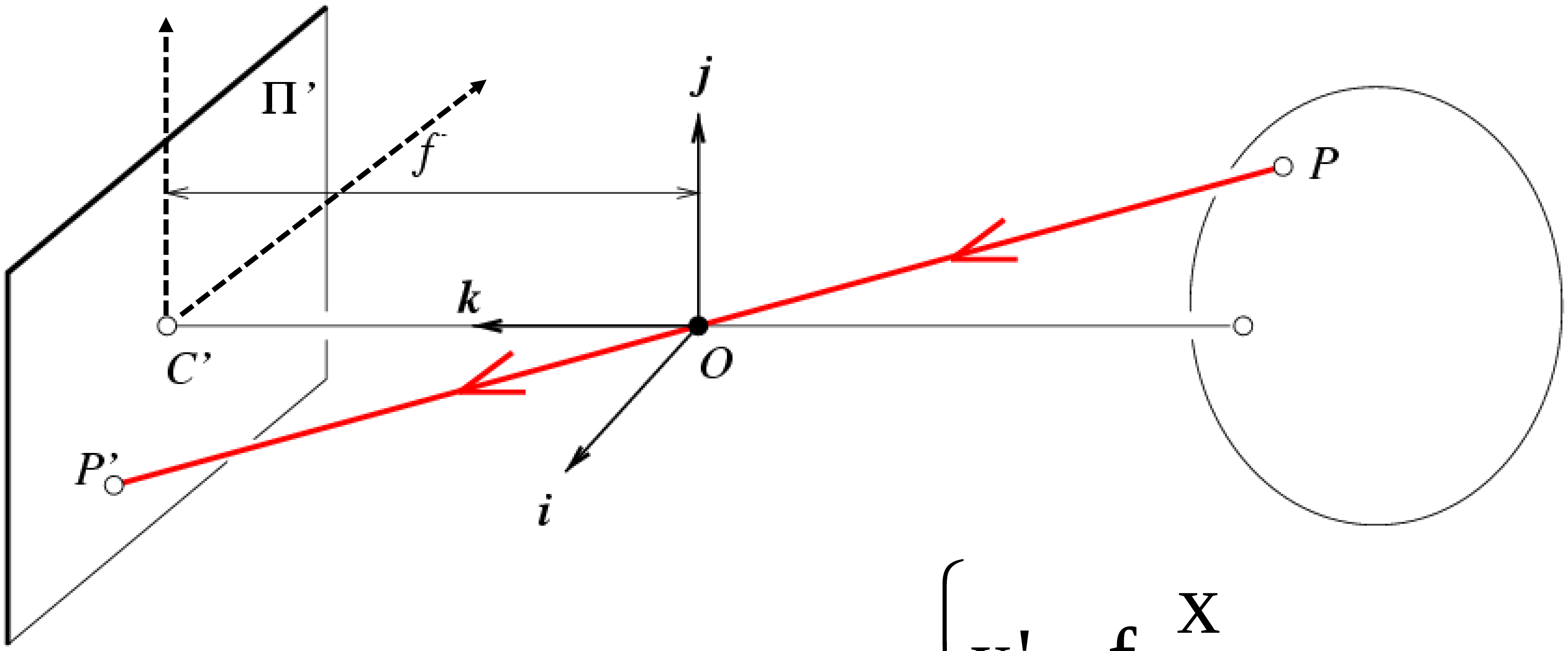
Al-Kindi (c. 801–873)
Ibn al-Haitham
(965-1040)

Pinhole camera



f = focal length
o = aperture = pinhole = center of the camera

Pinhole camera

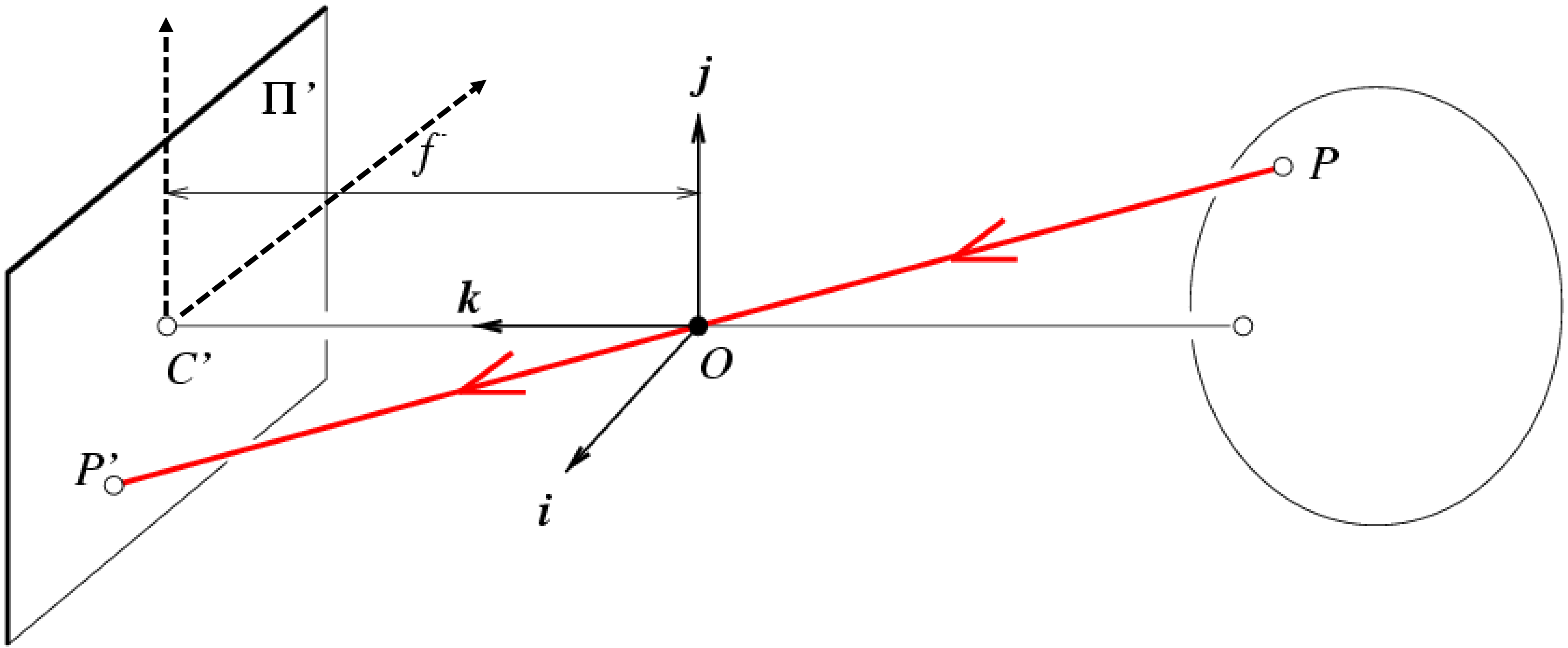


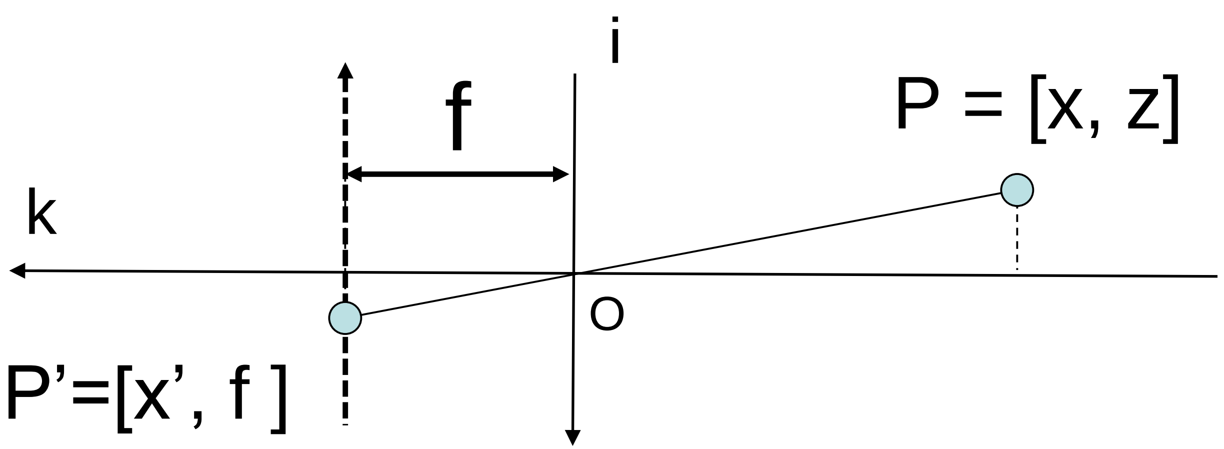
$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{cases} x' = f \dfrac{x}{z} \\ y' = f \dfrac{y}{z} \end{cases}$$

Derived using similar triangles

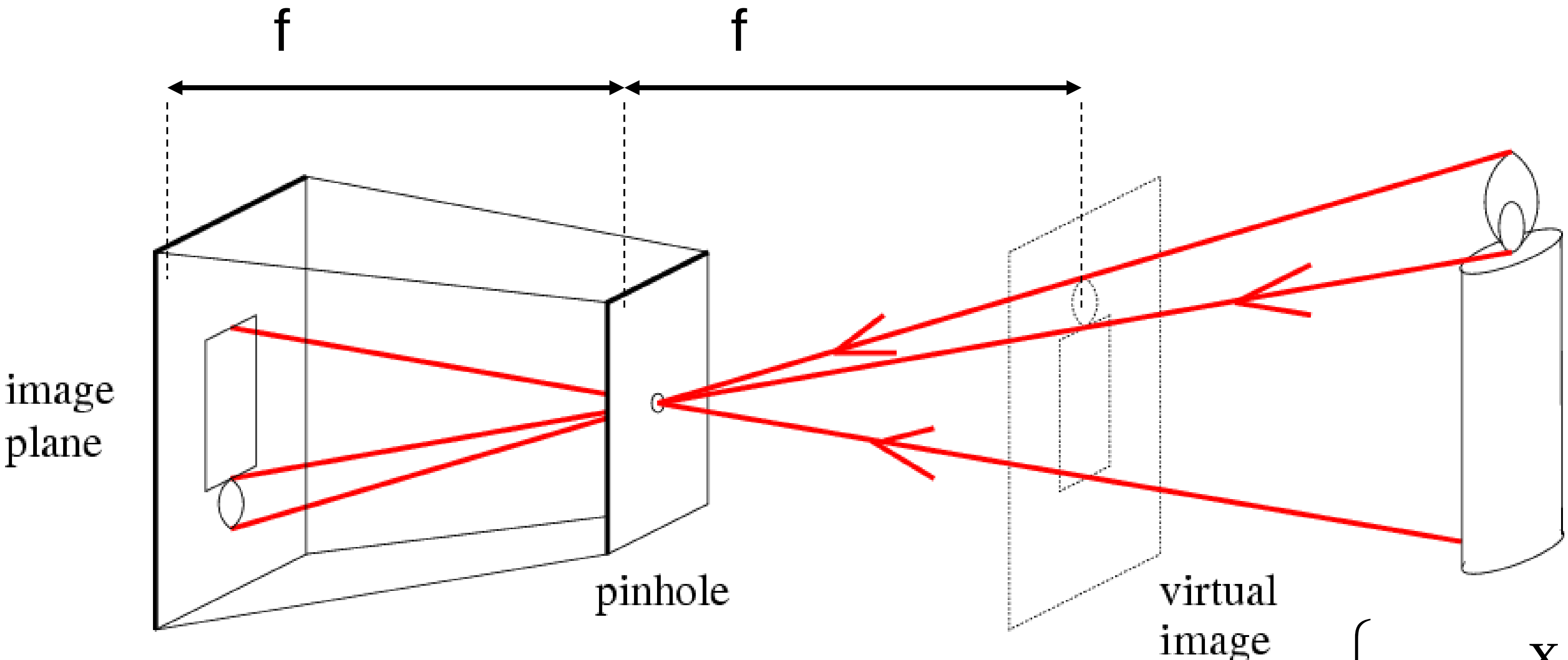
Pinhole camera





$$\frac{x'}{f} = \frac{x}{z}$$

Pinhole camera



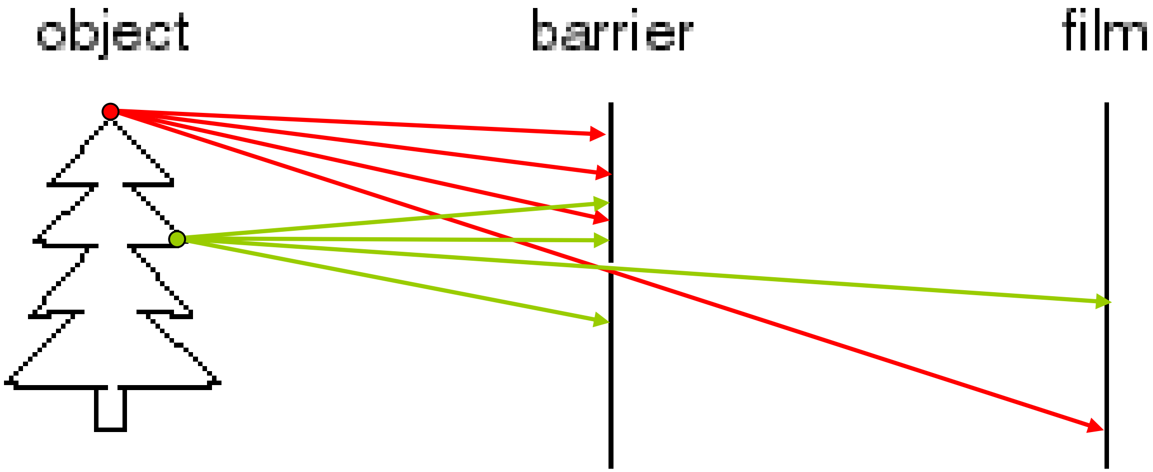
$$\begin{cases} x' = f\,\dfrac{x}{z} \\ y' = f\,\dfrac{y}{z} \end{cases}$$

Common to draw image plane *in front* of the focal point.

What's the transformation between these 2 planes?

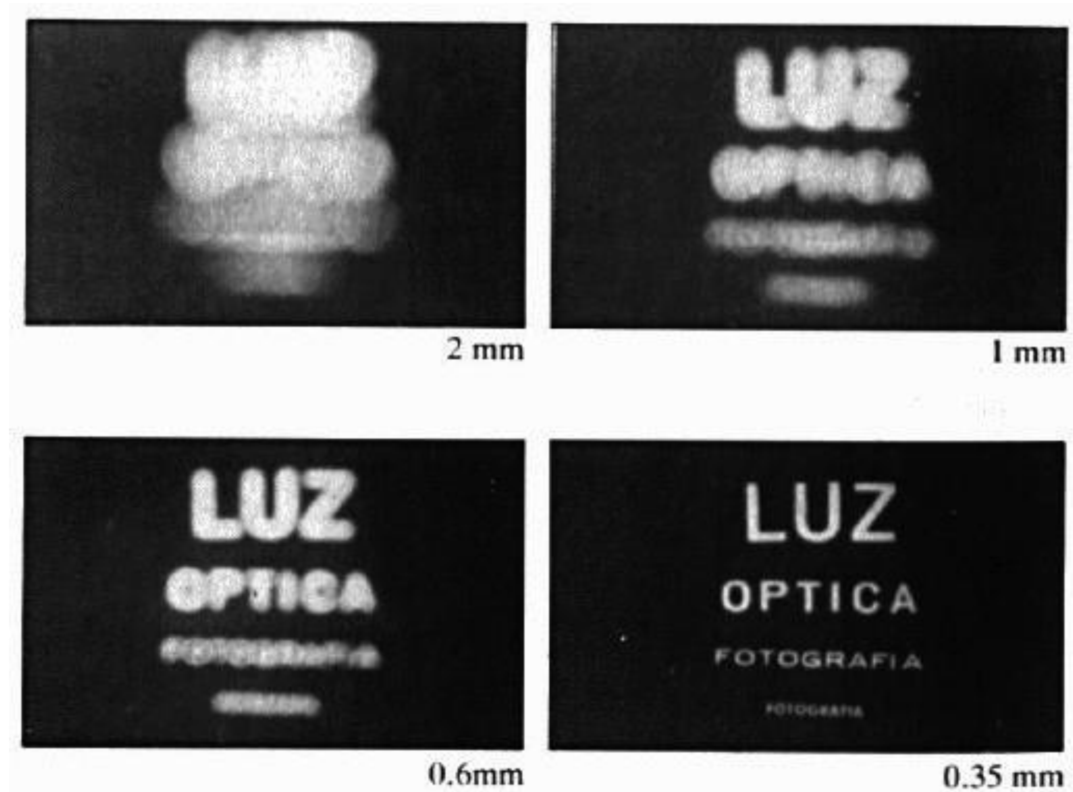
Pinhole camera

Is the size of the aperture important?



Shrinking aperture size

- Rays are mixed up

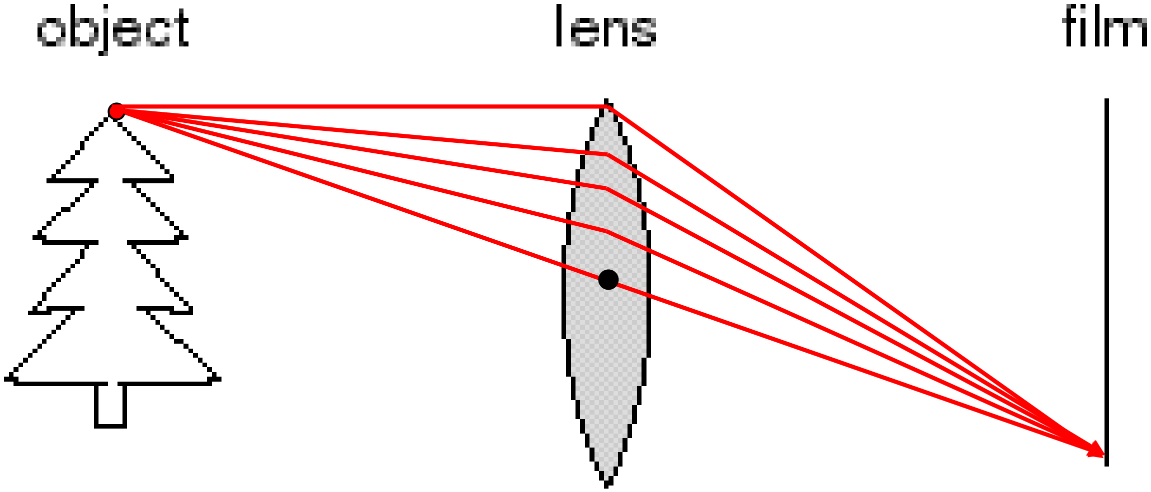


-Why the aperture cannot be too small?

-Less light passes through
-Diffraction effect

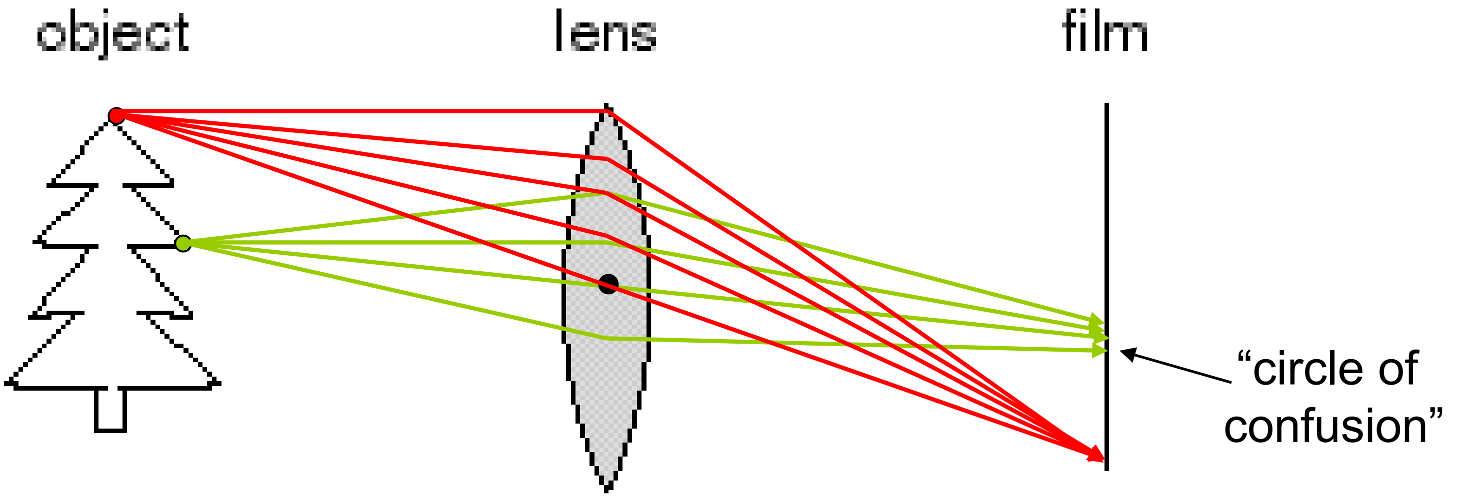
Adding lenses!

Cameras & Lenses



- A lens focuses light onto the film

Cameras & Lenses



- A lens focuses light onto the film
  - There is a specific distance at which objects are “in focus”
  - Related to the concept of depth of field

Cameras & Lenses

- A lens focuses light onto the film
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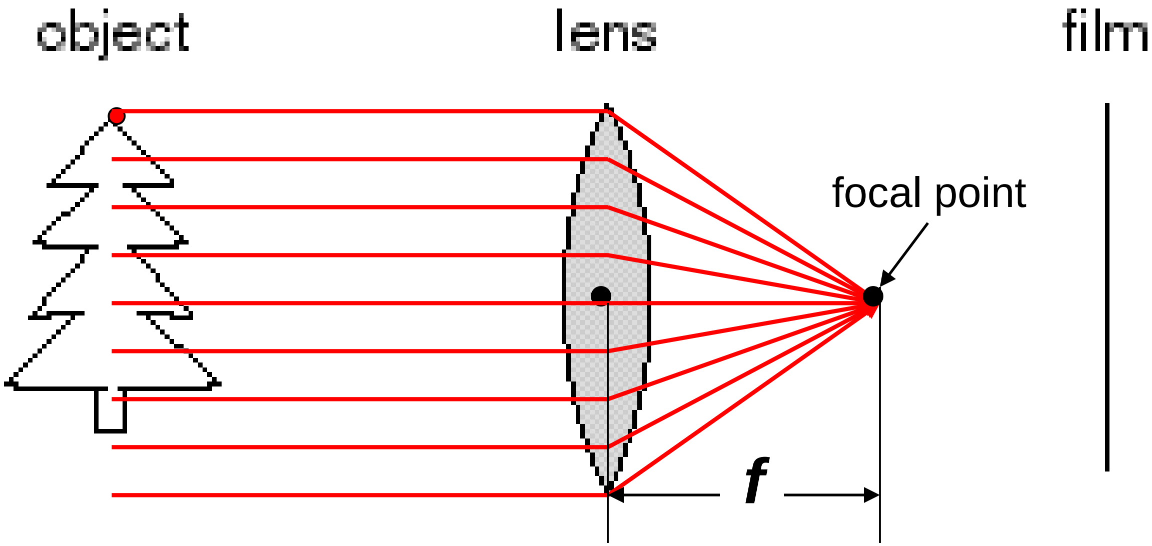
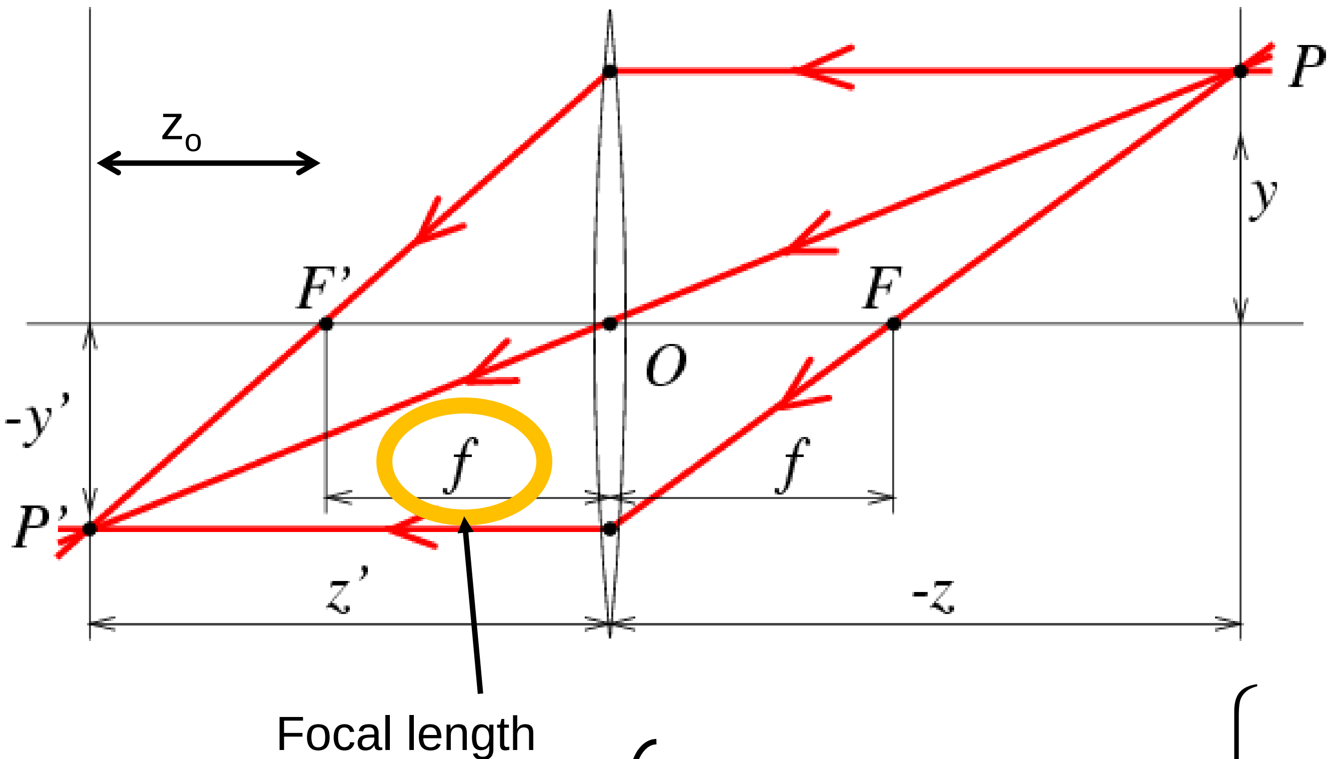
Cameras & Lenses



- A lens focuses light onto the film
  - All parallel rays converge to one point on a plane located at the *focal length f*
  - Rays passing through the center are not deviated

Thin Lenses

For details see lecture on cameras in CS131A



$$z' = f + z_o$$

$$f = \frac{R}{2(n-1)}$$

Snell's law:

$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$

Small angles:

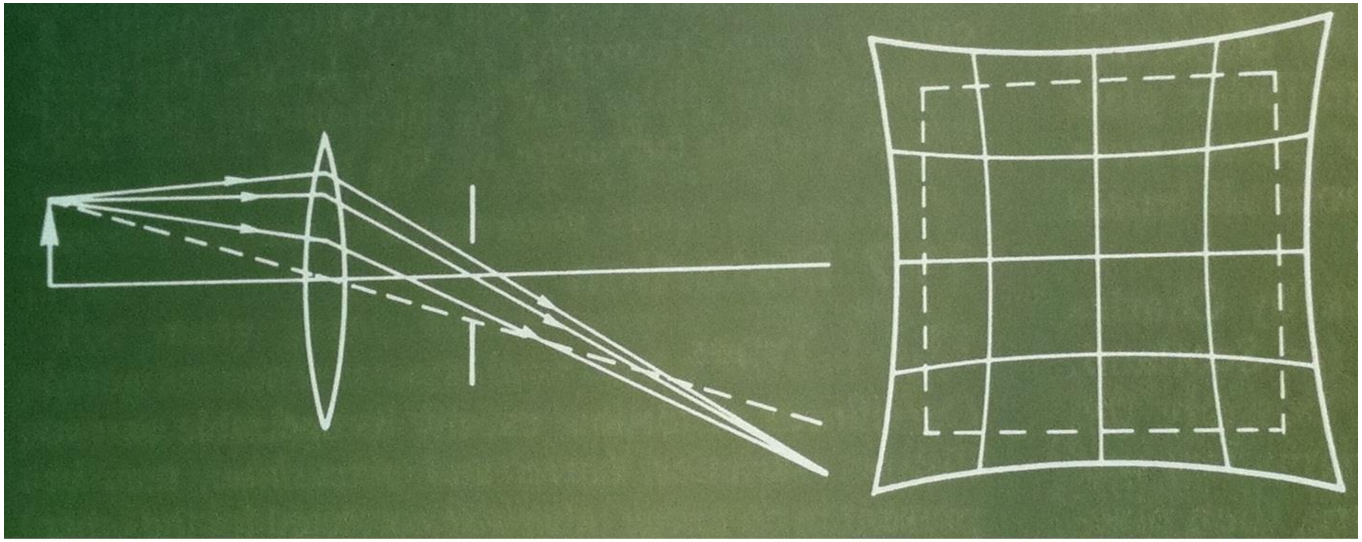
$n_1 \, \alpha_1 \approx n_2 \, \alpha_2$

$n_1 = \text{n}$ (lens)

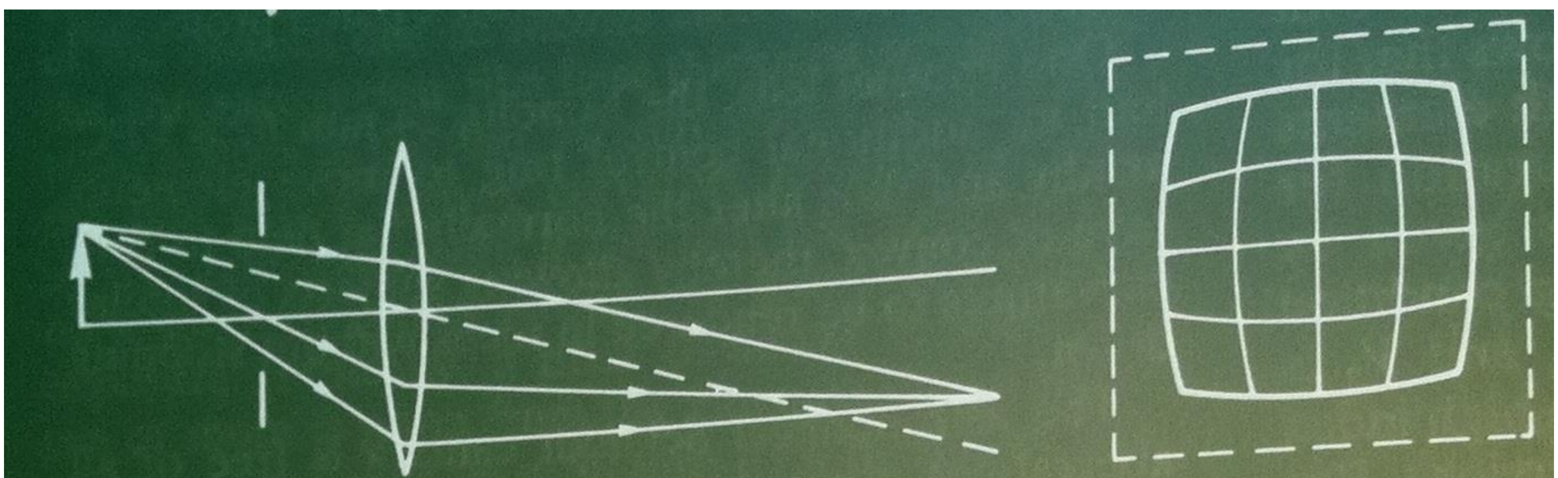
$n_1 = 1$ (air)

$$\begin{cases} x' = z' \dfrac{x}{z} \\ y' = z' \dfrac{y}{z} \end{cases}$$

Issues with lenses: Radial Distortion



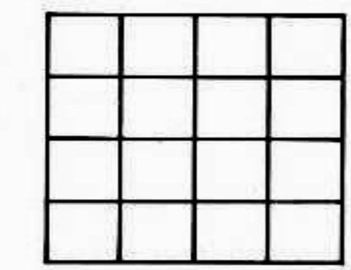
Pin cushion



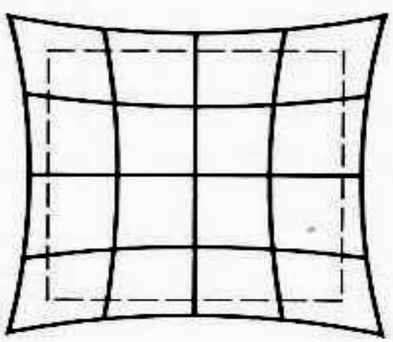
Barrel (fisheye lens)

Issues with lenses: Radial Distortion

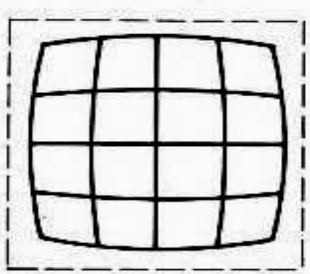
– Deviations are most noticeable for rays that pass through the edge of the lens



No distortion



Pin cushion



Barrel (fisheye lens)

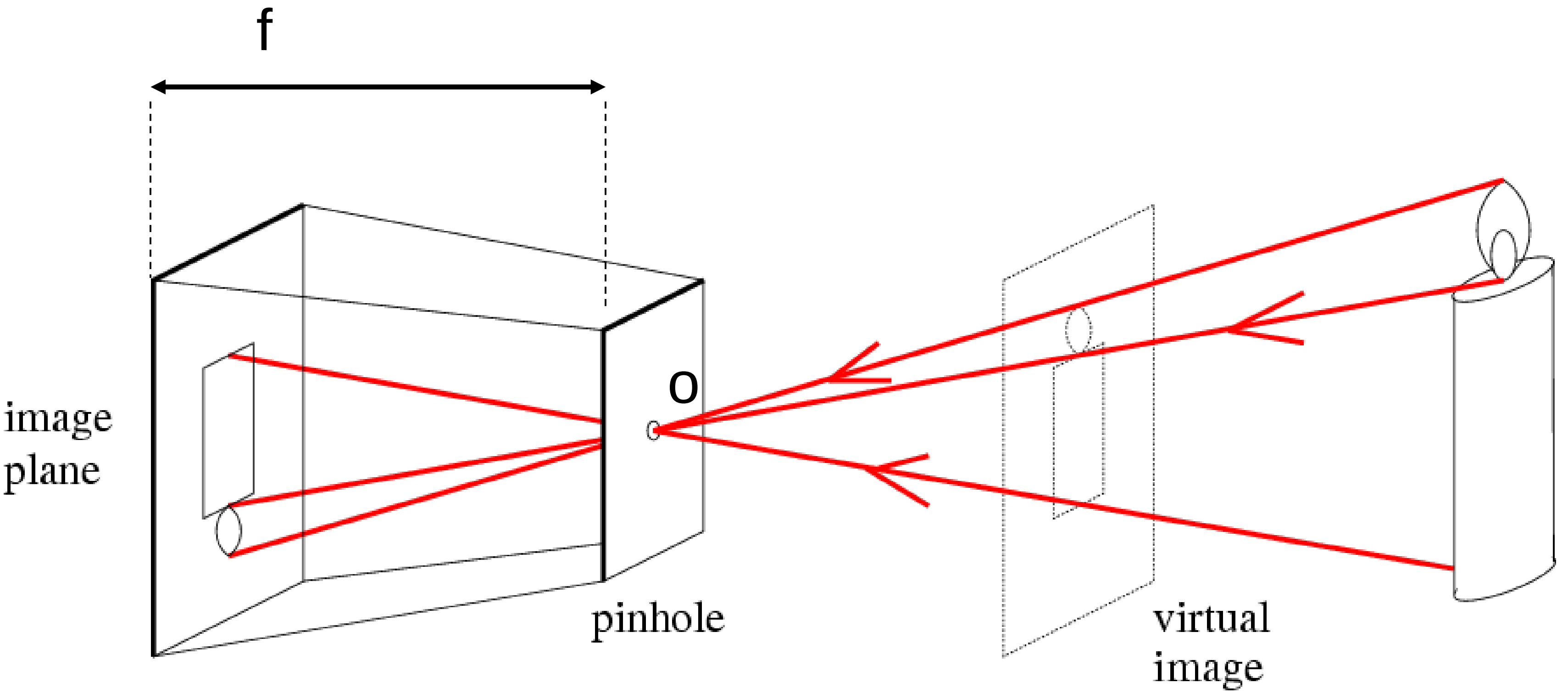
Image magnification decreases with distance from the optical axis

Lecture 2

Camera Models

- Pinhole cameras
- Cameras & lenses
- The geometry of pinhole cameras
  - Intrinsic
  - Extrinsic
- Other camera models

Pinhole camera

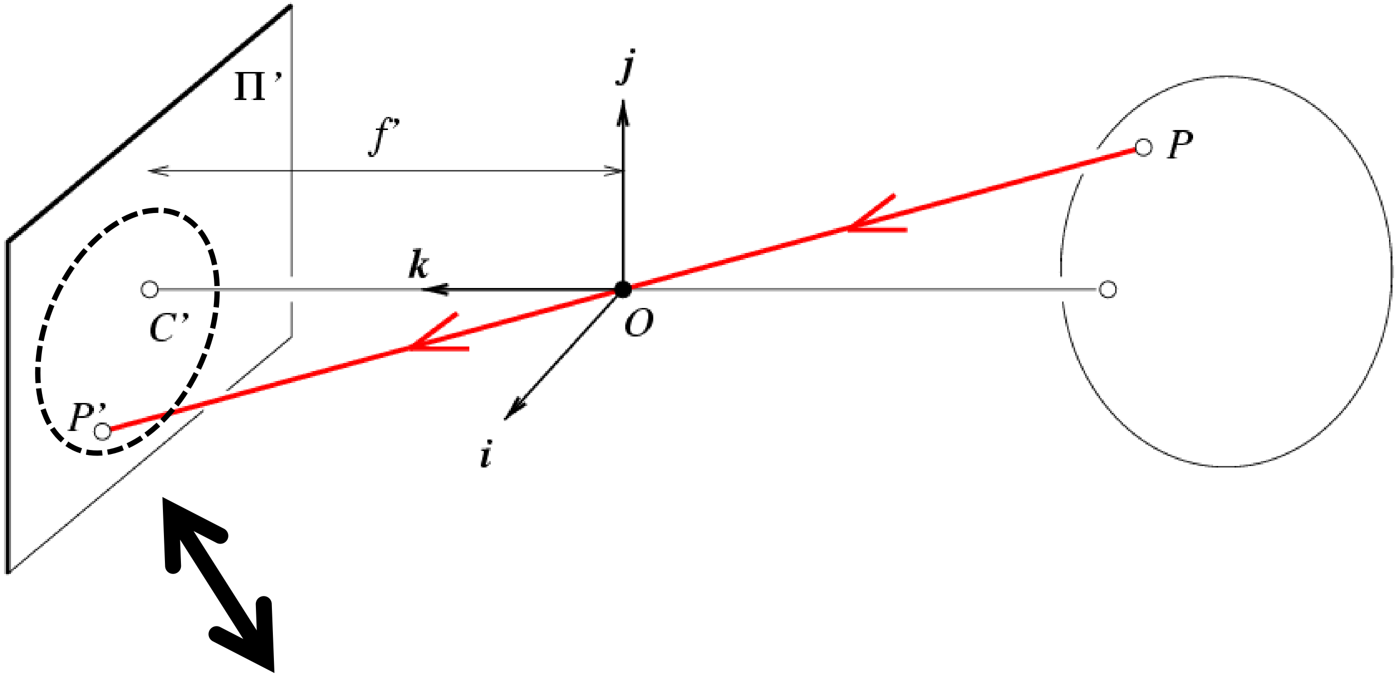


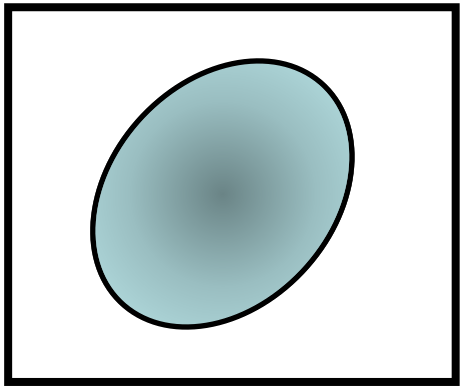
$$(x, y, z) \rightarrow (f\,\frac{x}{z}, f\,\frac{y}{z})$$

f = focal length
o = center of the camera

$$\Re^3 \xrightarrow{E} \Re^2$$

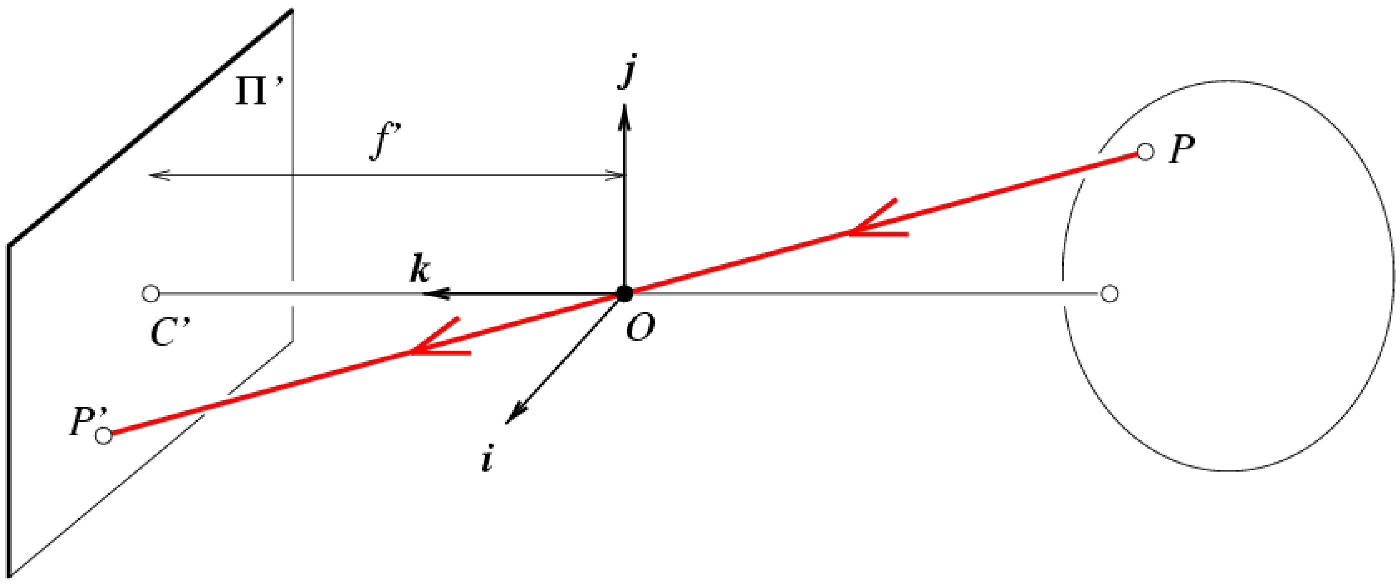
From retina plane to images

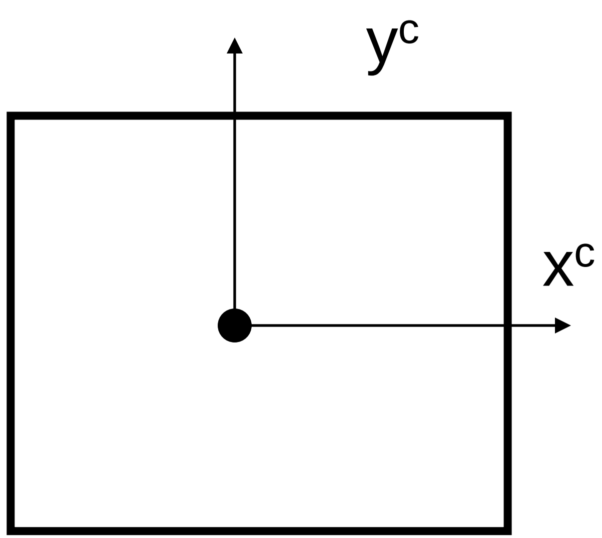




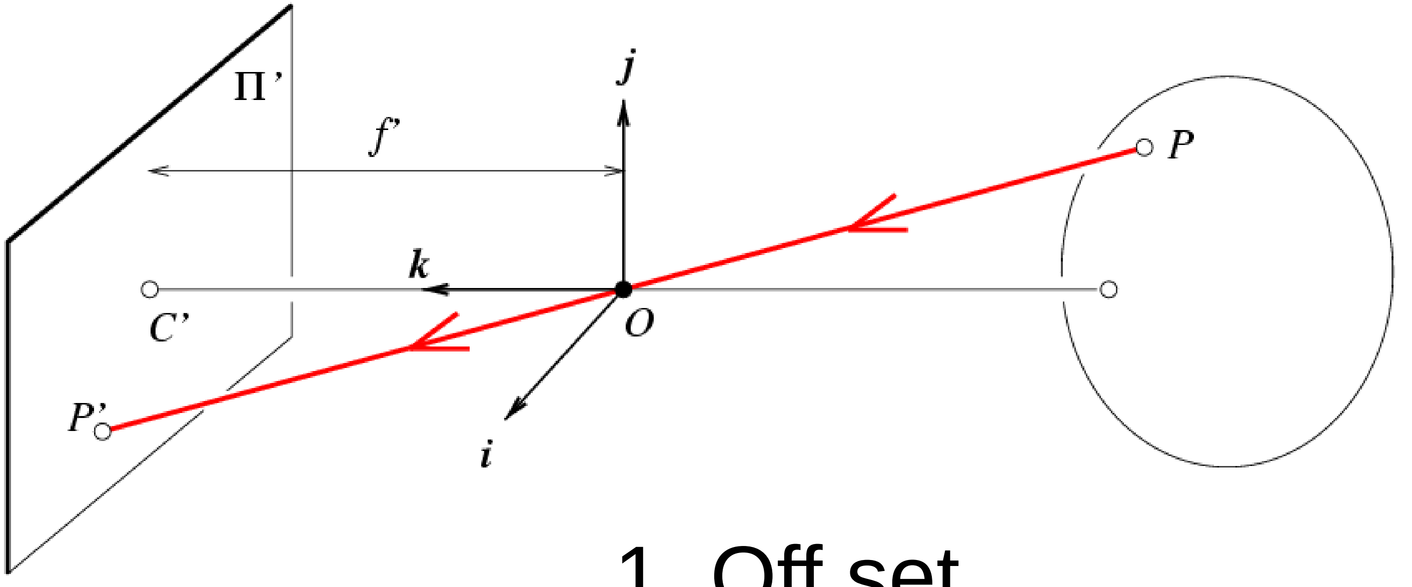
Pixels, bottom-left coordinate systems

Coordinate systems

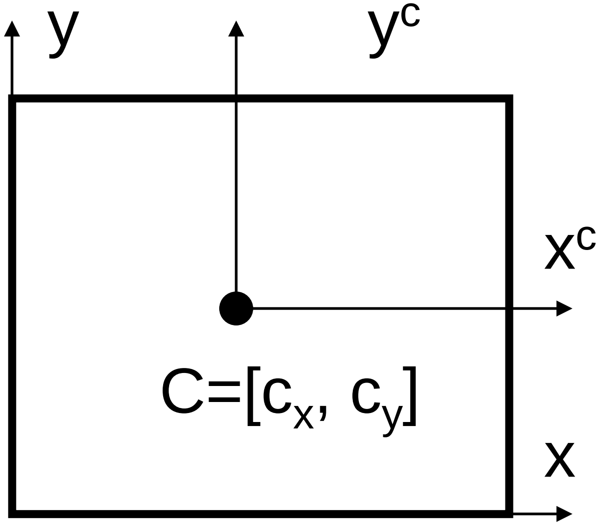




Converting to pixels

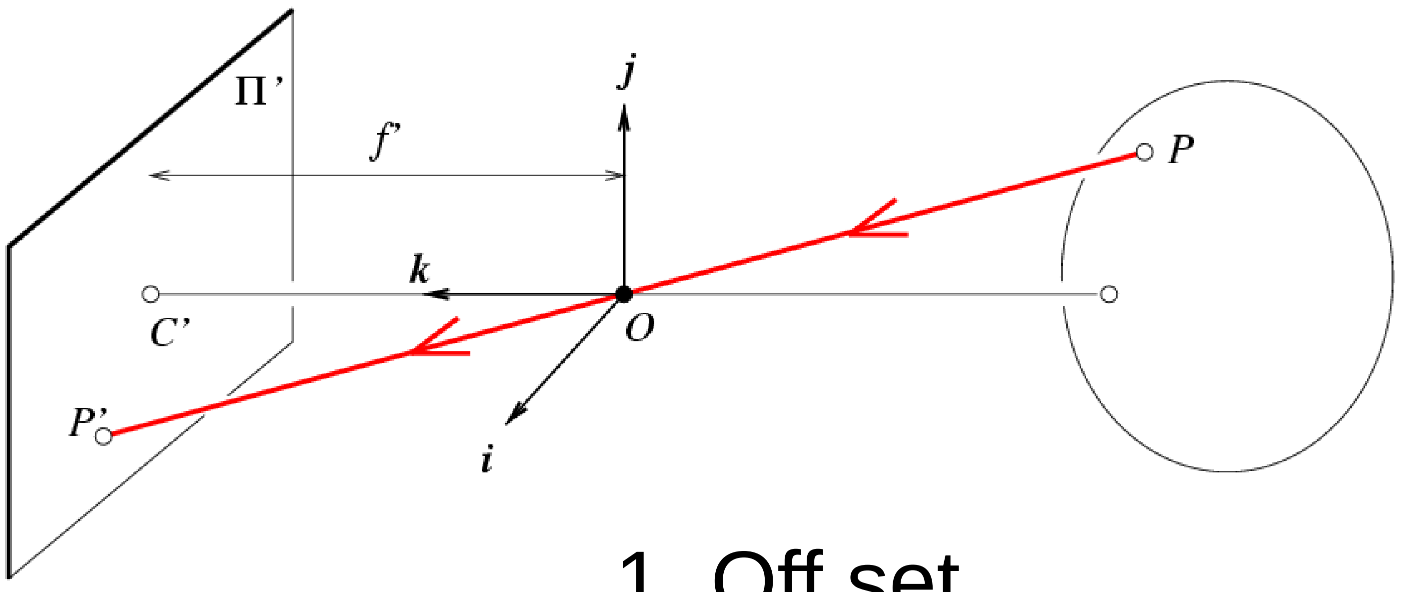


1. Off set

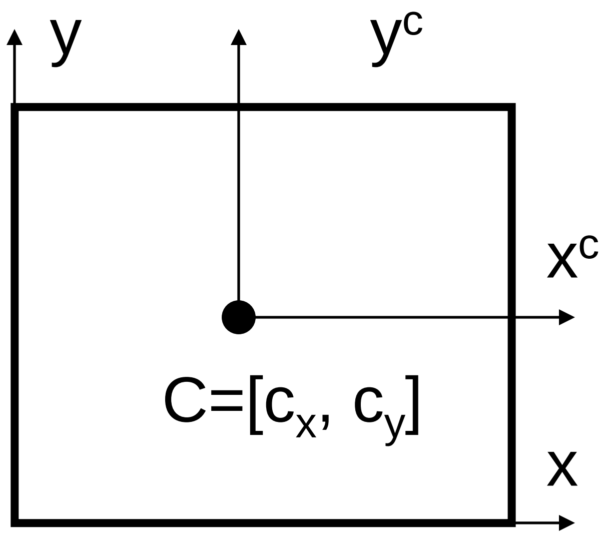


$$(x, y, z) \rightarrow (f\frac{x}{z} + c_x, f\frac{y}{z} + c_y)$$

Converting to pixels



1. Off set
2. From metric to pixels



$$(x, y, z) \rightarrow (\underbrace{f\,k}_{\alpha}\frac{x}{z} + c_x, \underbrace{f\,l}_{\beta}\frac{y}{z} + c_y)$$

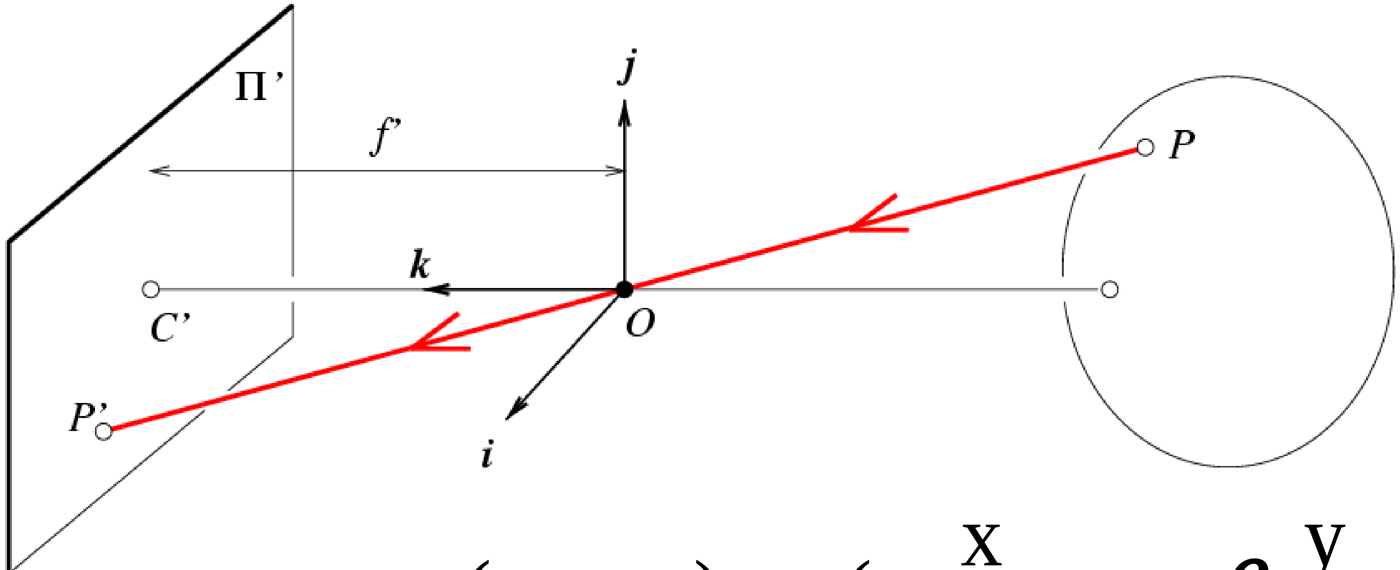
Units: k,l : pixel/m

f : m

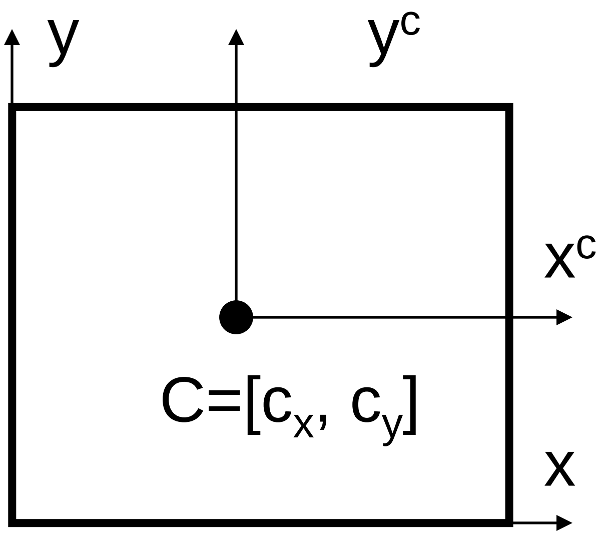
Non-square pixels

α, β : pixel

Converting to pixels



$$(x, y, z) \rightarrow (\boldsymbol{\alpha}\frac{x}{z} + c_x, \boldsymbol{\beta}\frac{y}{z} + c_y)$$



- Matrix form?

A related question:

- Is this a linear transformation?

$$(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$$

Is this a linear transformation?

No — division by z is nonlinear

How to make it linear?

Homogeneous coordinates

For details see lecture on transformations in CS131A

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

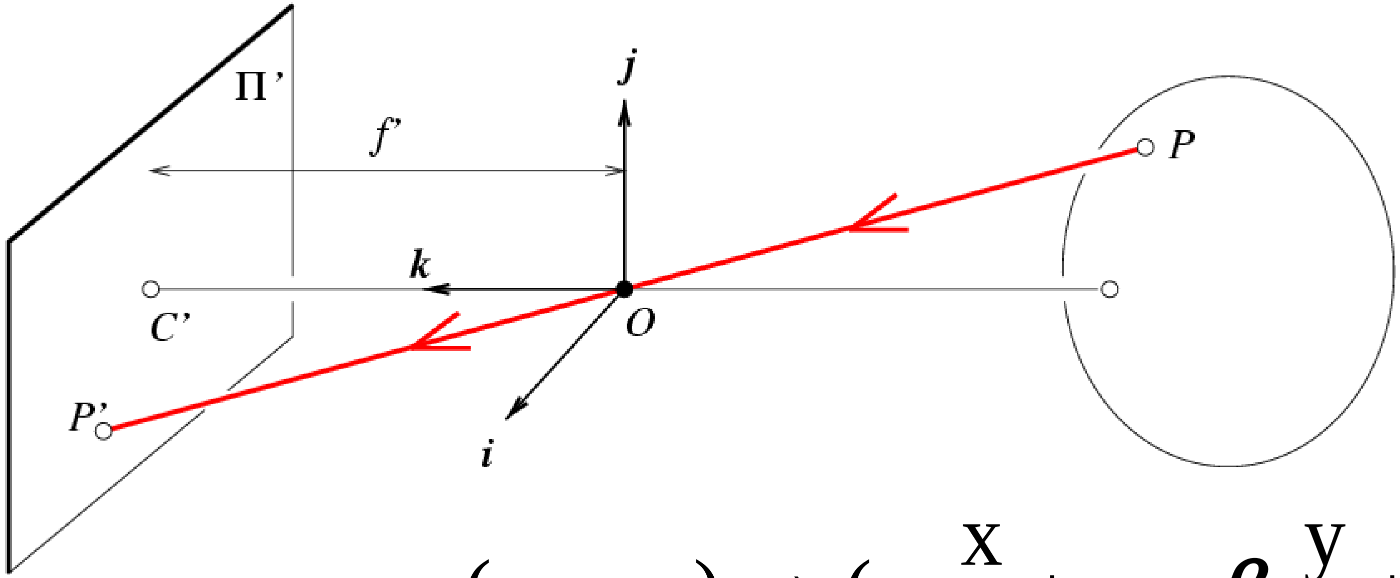
homogeneous scene coordinates

- Converting *from* homogeneous coordinates

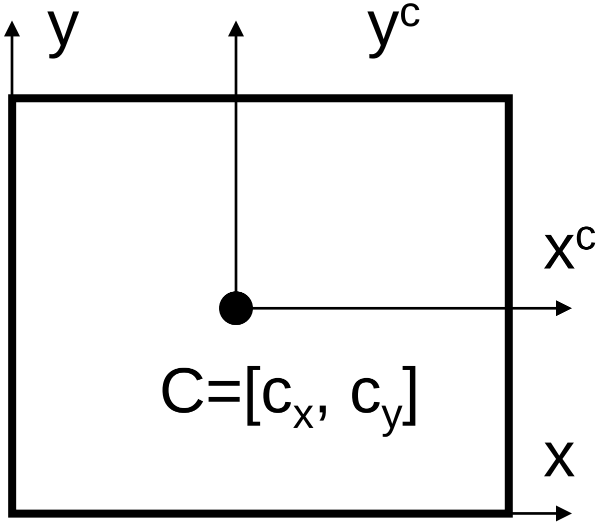
$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Camera Matrix



$$(x, y, z) \rightarrow (\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y)$$



$$X' = \begin{bmatrix} \alpha\, x + c_x z \\ \beta\, y + c_y z \\ z \end{bmatrix} = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Perspective Projection Transformation

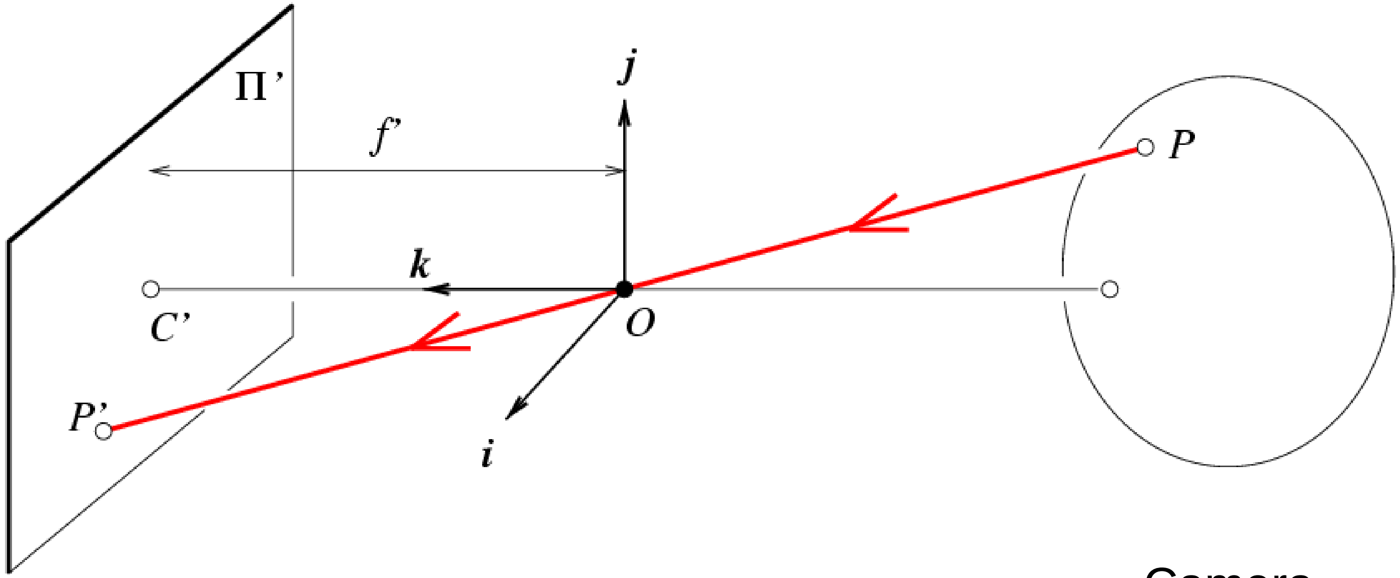
$$X' = \begin{bmatrix} f\,x \\ f\,y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{M} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$X' = M\,X$$

$$\Re^4 \xrightarrow{H} \Re^3$$

$$X_i' = \begin{bmatrix} f\,\dfrac{x}{z} \\ f\,\dfrac{y}{z} \end{bmatrix}$$

Camera Matrix



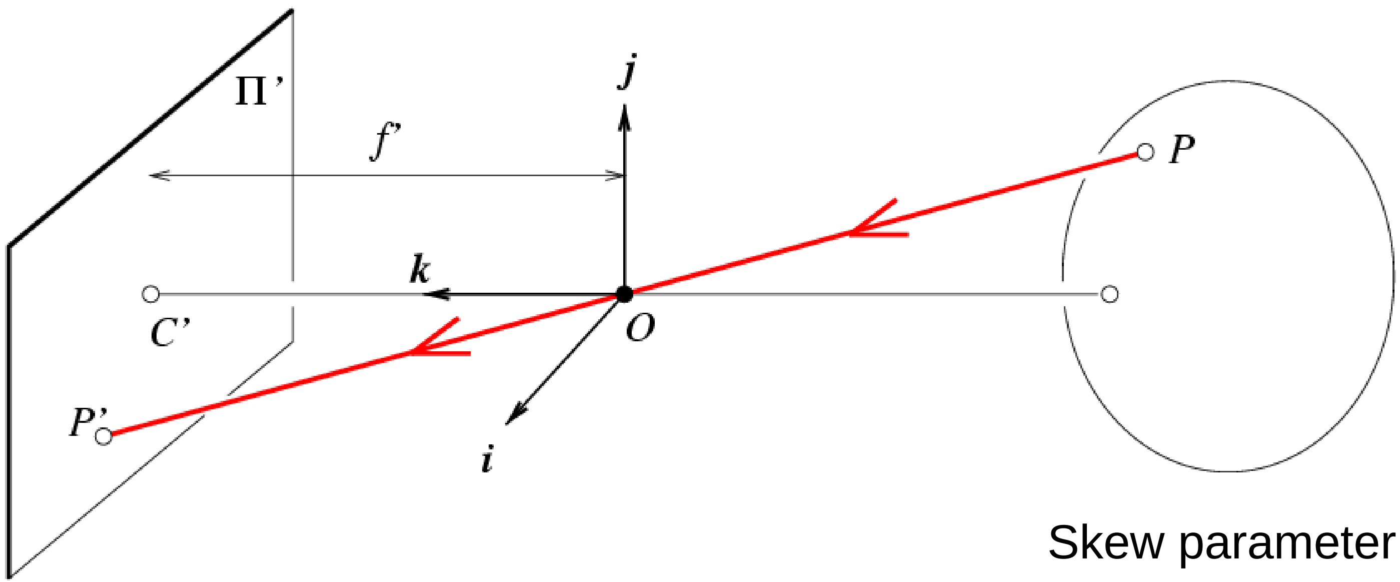
Camera matrix K

$$X' = M\,X$$

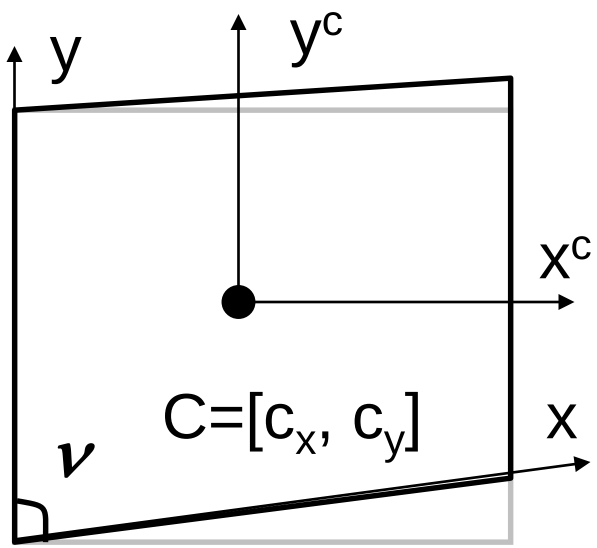
$$= K\begin{bmatrix} I & 0 \end{bmatrix} X$$

$$X' = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Finite projective cameras



Skew parameter



$$X' = \begin{bmatrix} \alpha & s & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

K has 5 degrees of freedom!

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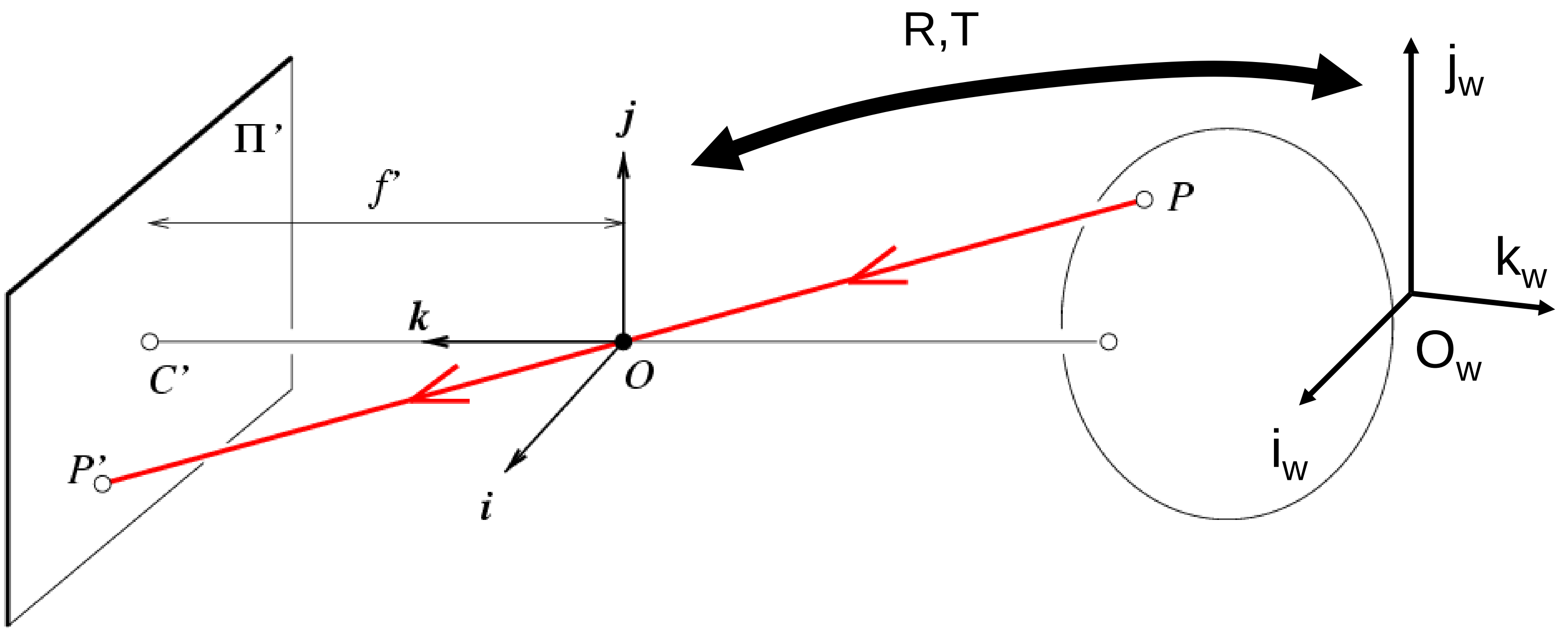
World reference system

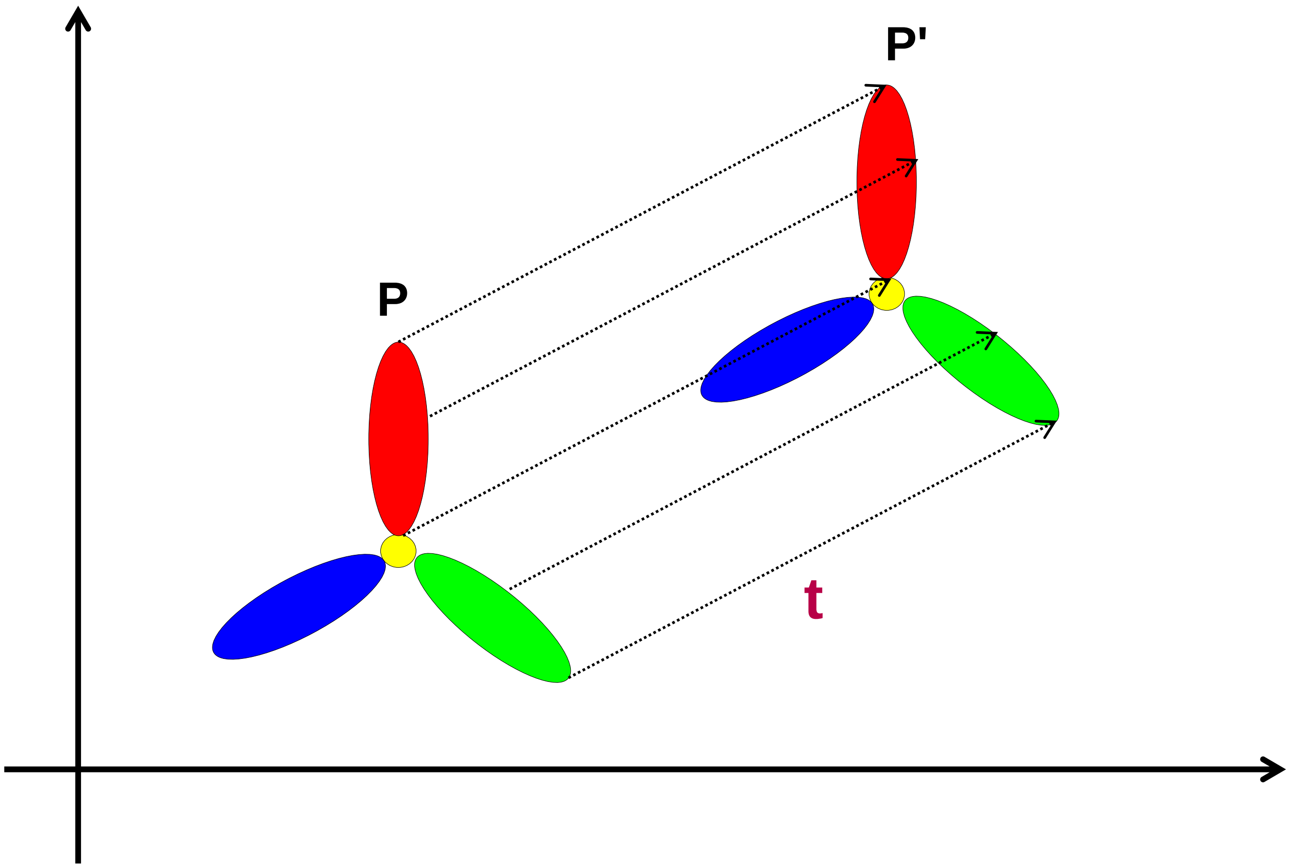


•The mapping so far is defined within the camera reference system

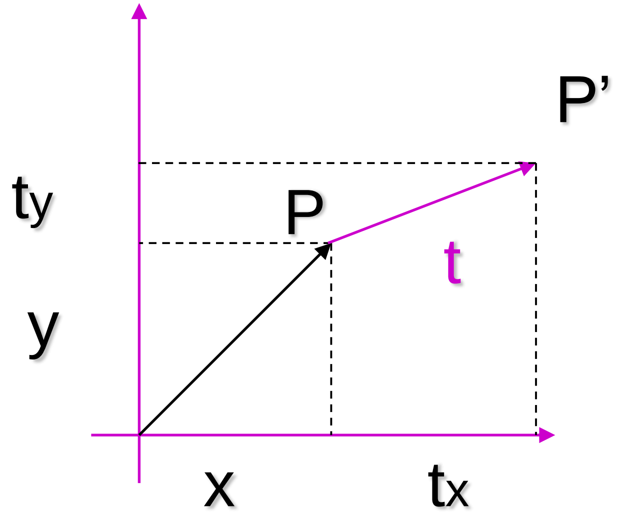
• What if an object is represented in the world reference system

- For details see lecture on transformations in CS131A
- See also TA session on Friday

2D Translation



2D Translation Equation

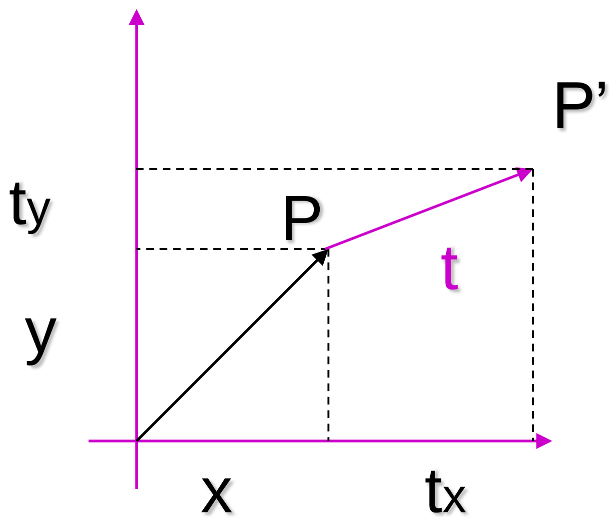


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = \mathbf{P} + \mathbf{t} = (\mathrm{x} + \mathrm{t}_{\mathrm{x}}, \mathrm{y} + \mathrm{t}_{\mathrm{y}})$$

2D Translation using Homogeneous Coordinates



$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

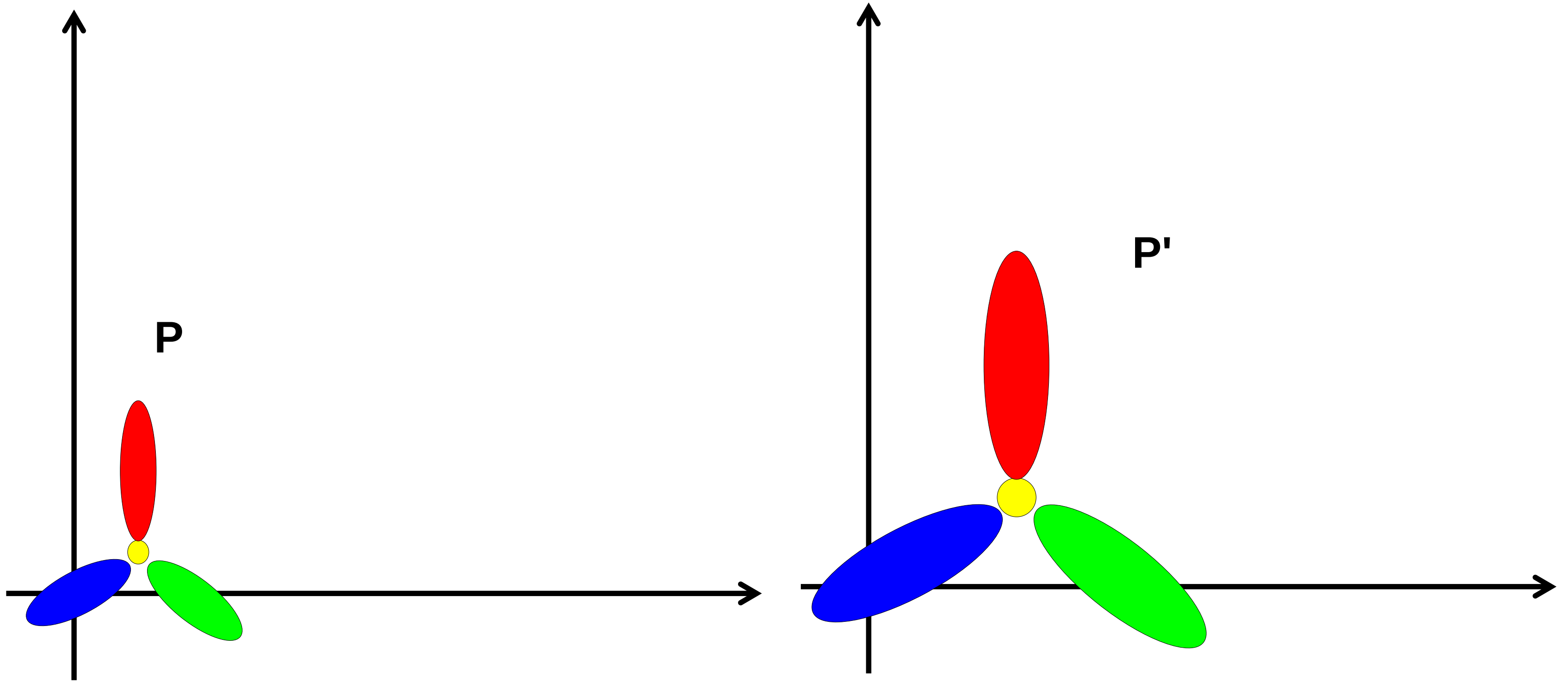
$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

t P

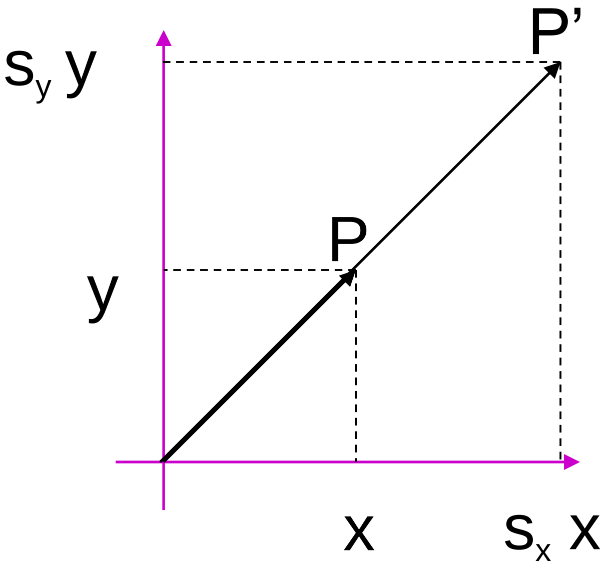
$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}$$

Scaling



Scaling Equation



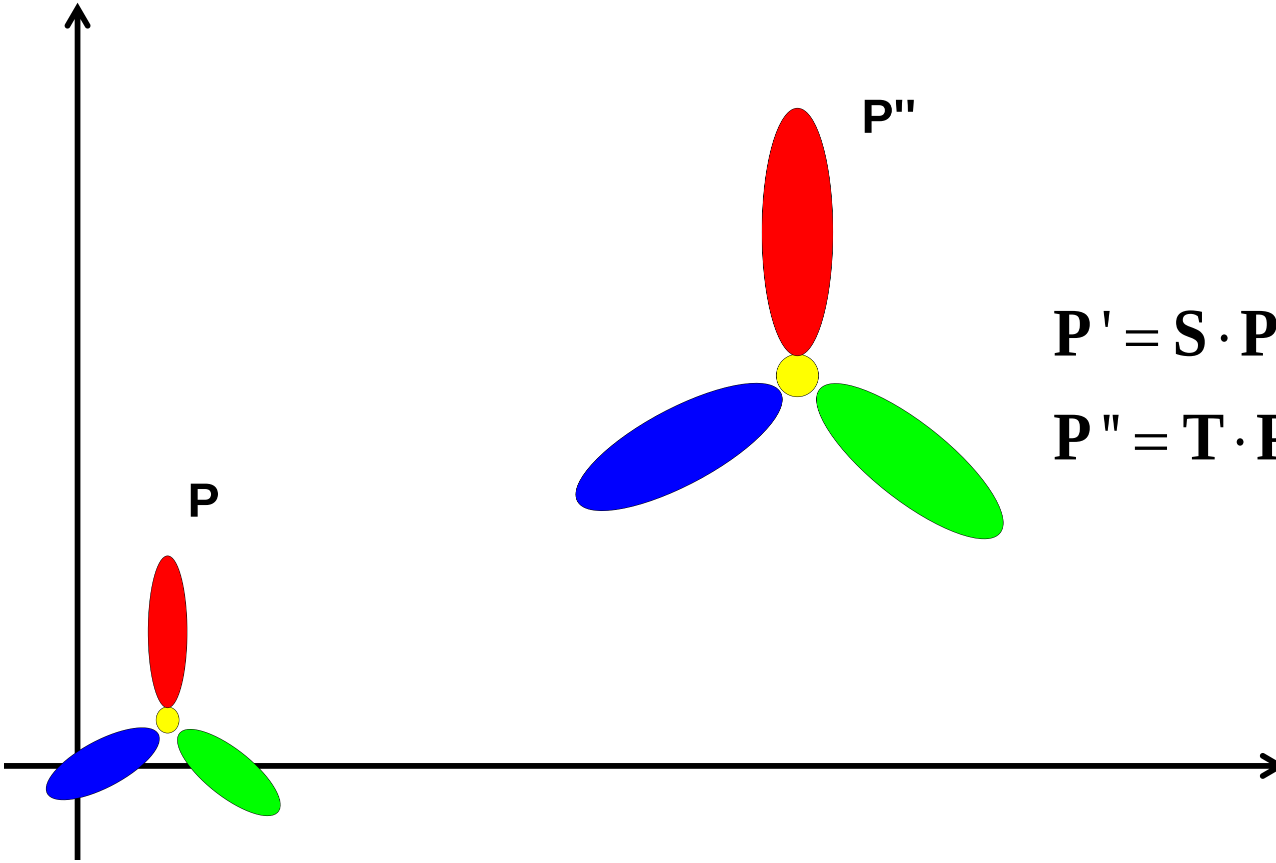
$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (s_x x, s_y y)$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}' & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

Scaling & Translating



$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

$$\mathbf{P}'' = \mathbf{T} \cdot \mathbf{P}'$$

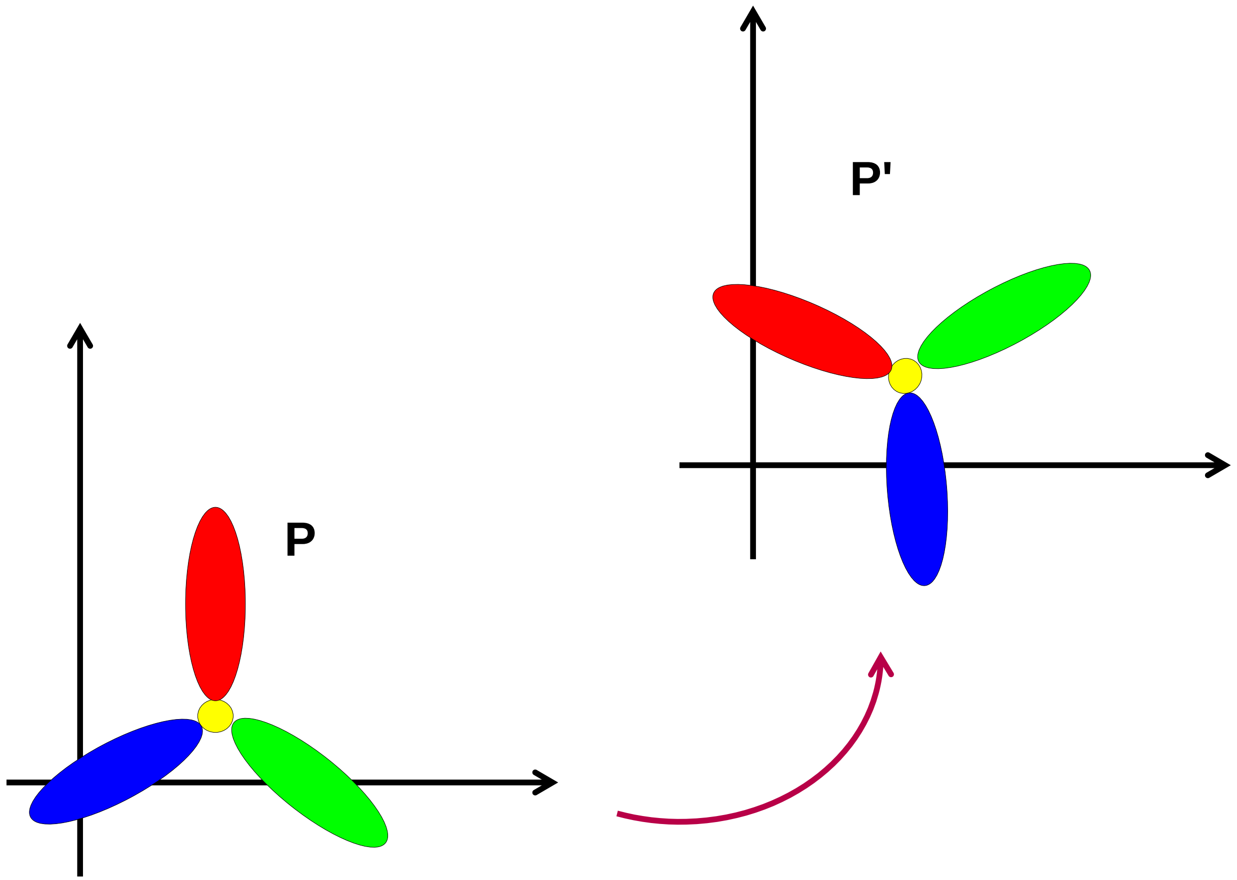
$$\mathbf{P}'' = \mathbf{T} \cdot \mathbf{P}' = \mathbf{T} \cdot (\mathbf{S} \cdot \mathbf{P}) = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \mathbf{A} \cdot \mathbf{P}$$

Scaling & Translating

$$\mathbf{P}'' = \mathbf{T}\cdot\mathbf{S}\cdot\mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

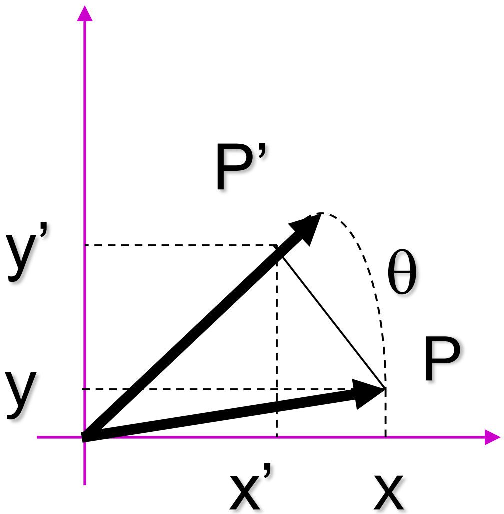
$$= \underbrace{\begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

Rotation



Rotation Equations

- Counter-clockwise rotation by an angle θ



$$x' = \cos\theta \; x - \sin\theta \; y$$

$$y' = \cos\theta \; y + \sin\theta \; x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R} \; \mathbf{P}$$

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2 ⟹ 4 elements

Note: R is an orthogonal matrix and satisfies many interesting properties:

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

Rotation + Scale + Translation

$$\mathbf{P}' = (\mathbf{T} \cdot \mathbf{R} \cdot \mathbf{S}) \cdot \mathbf{P}$$

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{R} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

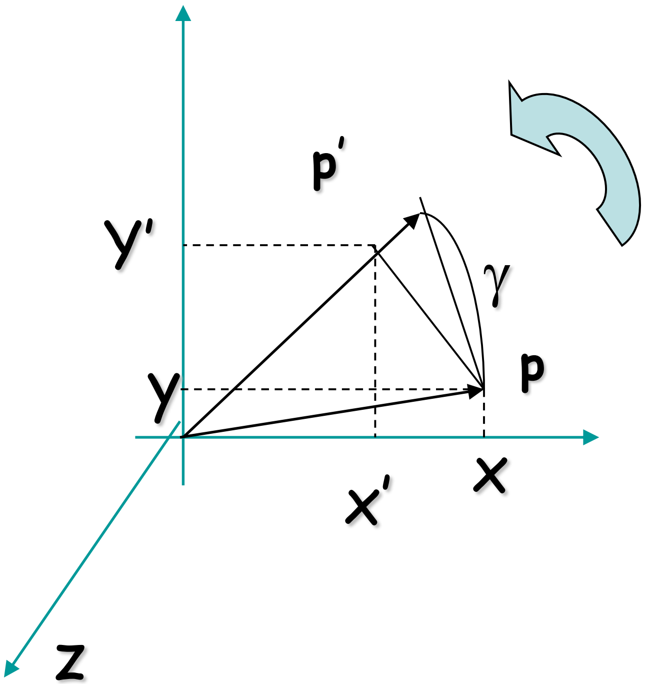
$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}' & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} \mathbf{R}'\mathbf{S} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

If $s_x = s_y$, this is a similarity transformation!

3D Rotation of Points

Rotation around the coordinate axes, counter-clockwise:

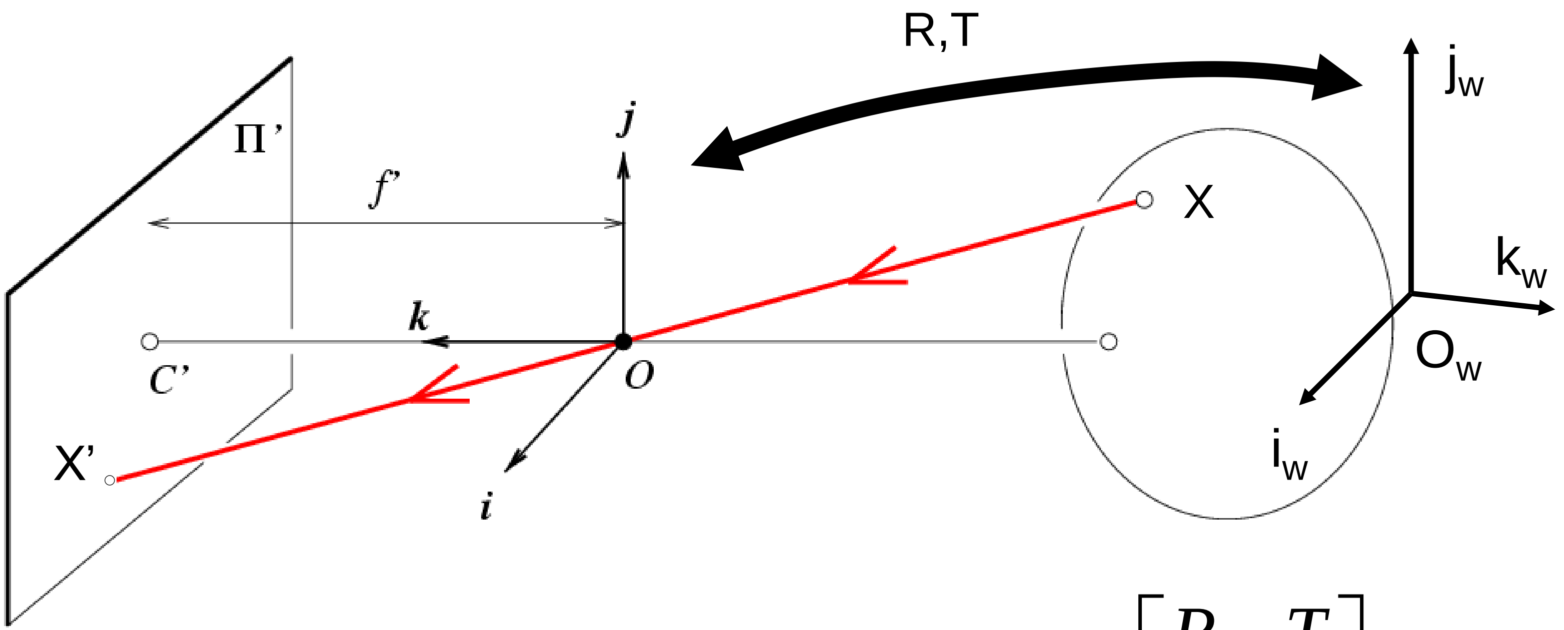


$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

World reference system



In 4D homogeneous coordinates: $X = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4\times4} X_w$

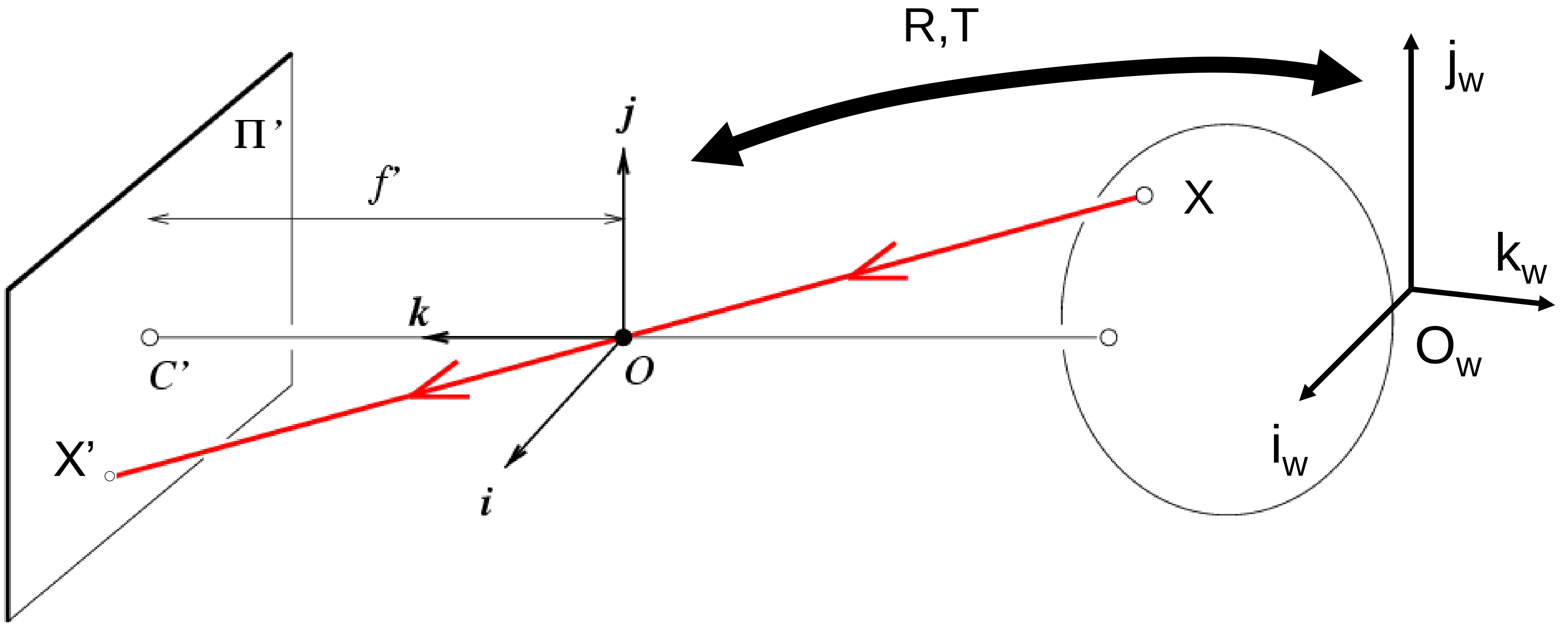
Internal parameters

External parameters

$$X' = K[I \quad 0]\, X = K\,[I \quad 0] \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4\times4} X_w = K[R \quad T]\, X_w$$

M

Projective cameras

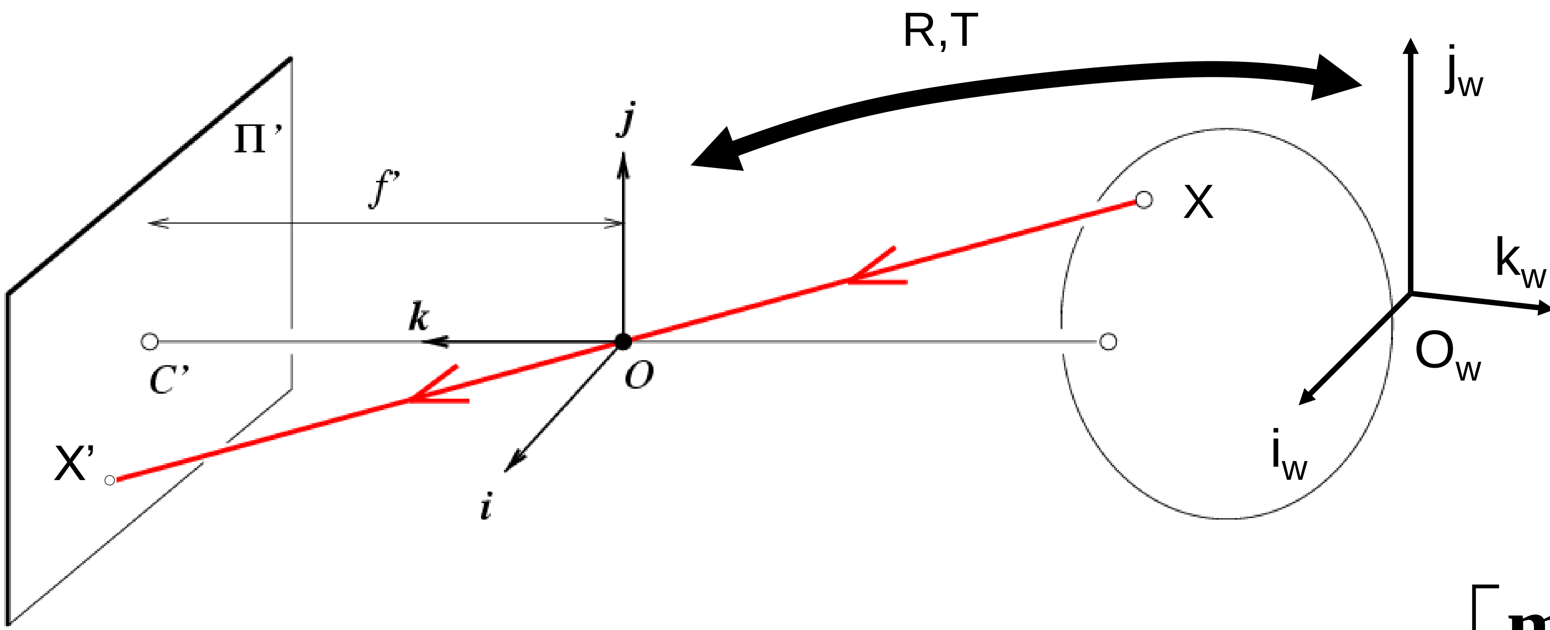


$$X'_{3\times1} = M_{3x4}\ X_w = K_{3\times3}\begin{bmatrix} R & T \end{bmatrix}_{3\times4}\ X_{w\,4\times1} \qquad K = \begin{bmatrix} \alpha & s & c_x \\ 0 & \beta & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

How many degrees of freedom?

5 + 3 + 3 =11!

Projective cameras



$$X'_{3\times1} = \mathrm{M}\ X_w = K_{3\times3}\begin{bmatrix} R & T \end{bmatrix}_{3\times4} X_{w\,4\times1} \qquad M = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} X_{\mathbf{w}} = \begin{bmatrix} \mathbf{m}_1 X_{\mathbf{w}} \\ \mathbf{m}_2 X_{\mathbf{w}} \\ \mathbf{m}_3 X_{\mathbf{w}} \end{bmatrix} \quad \overset{E}{\rightarrow} \left(\frac{\mathbf{m}_1 X_w}{\mathbf{m}_3 X_w}, \frac{\mathbf{m}_2 X_w}{\mathbf{m}_3 X_w}\right)$$

Theorem (Faugeras, 1993)

$$M = K\begin{bmatrix} R & T \end{bmatrix} = \begin{bmatrix} K\,R & K\,T \end{bmatrix} = \begin{bmatrix} A & b \end{bmatrix}$$

$$K = \begin{bmatrix} \alpha & s & c_x \\ 0 & \beta & c_y \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} \boldsymbol{\alpha} &= \mathrm{f\ k}; \\ \boldsymbol{\beta} &= \mathrm{f\ l} \end{aligned} \qquad A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}$$

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix is that $\mathrm{Det}(\mathcal{A}) \neq 0$.
- A necessary and sufficient condition for $\mathcal{M}$ to be a zero-skew perspective projection matrix is that $\mathrm{Det}(\mathcal{A}) \neq 0$ and

$$(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0.$$

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix with zero skew and unit aspect-ratio is that $\mathrm{Det}(\mathcal{A}) \neq 0$ and

$$\begin{cases} (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0, \\ (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3). \end{cases}$$

Properties of Projection

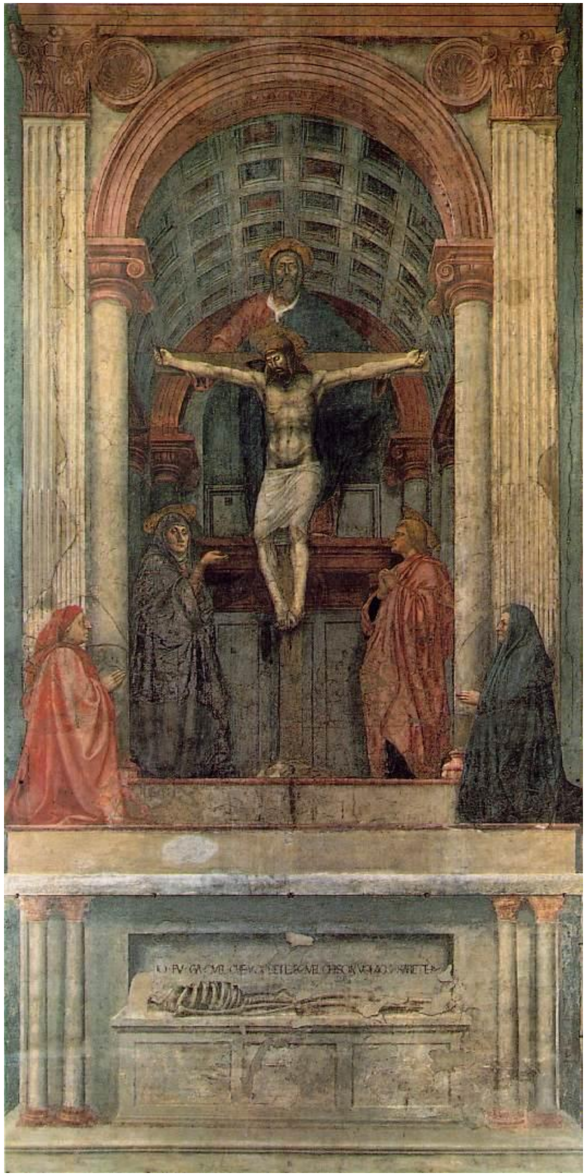
- Points project to points
- Lines project to lines
- Distant objects look smaller

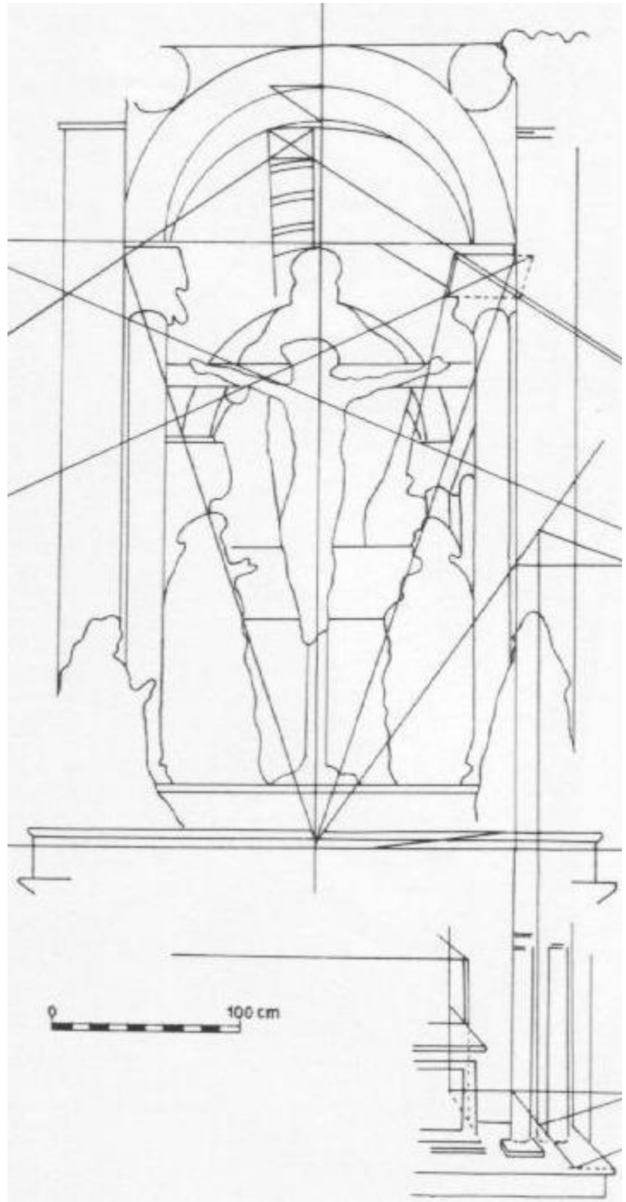
Properties of Projection

- Angles are not preserved
- Parallel lines meet!

Vanishing point

One-point perspective





- Masaccio, *Trinity*, Santa Maria Novella, Florence, 1425-28

Credit slide S. Lazebnik

Next lecture

- How to calibrate a camera?