Linear systems and Convolutions

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Linear Systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

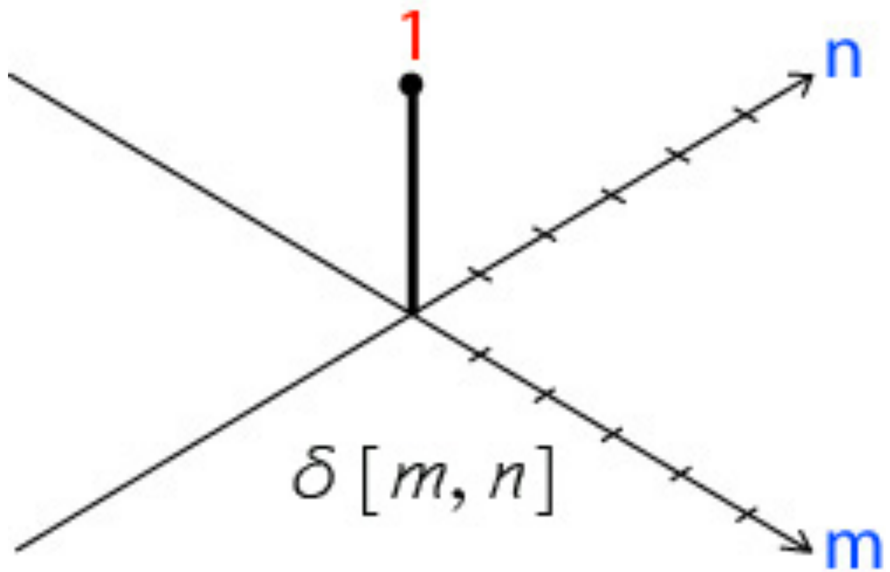
- Linear filtering:
  - Form a new image whose pixels are a weighted sum of original pixel values
  - Use the same set of weights at each point
- **S** is a linear system (function) iff it *S satisfies*

$$S[\alpha f_i[n,m] + \beta f_j[h,m]] = \alpha S[f_i[n,m]] + \beta S[f_j[h,m]]$$

superposition property

2D impulse function

- 1 at [0,0].
- 0 everywhere else



LSI (linear *shift invariant*) systems

Impulse response

$$\delta_2[n, m] \rightarrow \boxed{\mathcal{S}} \rightarrow h[n, m]$$

$$\delta_2[n-k, m-l] \rightarrow \boxed{\mathcal{S}\ (\mathrm{SI})} \rightarrow h[n-k, m-l]$$

Why are convolutions flipped?

Let's first represent f[0,0] as a sum of deltas:

$$f[0,0] = f[0,0]\times 1$$
$$= f[0,0]\times\delta[0,0]$$
$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f[k,l]\times\delta[k,l]$$

Or

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f[k,l]\times\delta[-k,-l]$$

Why are convolutions flipped?

Let's first represent f[1,1] as a sum of deltas:

$$
\begin{aligned}
f[1,1] &= f[1,1] \times 1 \\
&= f[1,1] \times \delta[0,0] \\
&= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \times \delta[1-k, 1-l]
\end{aligned}
$$

Why are convolutions flipped?

Now for the general case, let′s write
f[n, m] as a sum of deltas:

$$
\begin{aligned}
f[n,m] &= f[n,m] \times 1 \\
&= f[n,m] \times \delta[0,0] \\
&= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \times \delta[n-k, m-l]
\end{aligned}
$$

Why are convolutions flipped?

- Now, let's pass this function through a linear shift invariant (LSI) system:

$$f[n,m] \xrightarrow{S} S[f[n,m]]$$

$$S[f[n,m]] = S\left[\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \times \delta[n-k, m-l] \right]$$

Let's apply the superposition property here:

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l]\, S[\delta[n-k, m-l]]$$

Why are convolutions flipped?

Copying the equation from the previous slide:

S[f[n,m]] $= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l]\, S[\delta[n-k, m-l]]$

Finally, we know what happens when we pass a delta function through a system right? We can use that here:

S[f[n,m]] $= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l]\, h[n-k, m-l]$

And this is how we get our flipped convolution function