Lecture 8: Camera Models

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What we will learn today?

• Pinhole cameras
• Cameras & lenses
• The geometry of pinhole cameras
  • Projection matrix
  • Intrinsic parameters
  • Extrinsic parameters

Reading:
[FP] Chapters 1 – 3
[HZ] Chapter 6
What we will learn today?

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Reading:
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Camera and World Geometry

- How tall is this woman?
- How high is the camera?
- What is the camera rotation?
- What is the focal length of the camera?
- Which ball is closer?
How do we see the world?

- Let’s design a camera
  - Idea 1: put a piece of film in front of an object
  - Do we get a reasonable image?
• Add a barrier to block off most of the rays
  – This reduces blurring
  – The opening known as the aperture
Camera obscura: the pre-camera

- Known during classical period in China and Greece (e.g. Mo-Tsi, China, 470BC to 390BC)

Illustration of Camera Obscura

Freestanding camera obscura at UNC Chapel Hill

Photo by Seth Ilys

Slide credit: J. Hayes
Camera Obscura used for Tracing

Lens Based Camera Obscura, 1568

Slide credit: J. Hayes
**First Photograph**

Oldest surviving photograph
- Took 8 hours on pewter plate

Photograph of the first photograph

Joseph Niepce, 1826

Stored at UT Austin

Niepce later teamed up with Daguerre, who eventually created Daguerrotypes

Slide credit: J. Hayes
Dimensionality Reduction Machine (3D to 2D)

3D world

Point of observation

2D image

Figures © Stephen E. Palmer, 2002
Projection can be tricky...
Projection can be tricky...
Projective Geometry

What is lost?

• Length
Length is not preserved
Projective Geometry

What is lost?

• Length
• Angles
What is preserved?

• Straight lines are still straight
Vanishing points and lines

Parallel lines in the world intersect in the image at a “vanishing point”

Slide credit: J. Hayes
Vanishing points and lines

Vertical vanishing point (at infinity)

Vanishing point

Vanishing line

Vanishing point
Pinhole camera

\[
P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \rightarrow \quad P' = \begin{bmatrix} x' \\ y' \end{bmatrix}
\]

\[
\begin{align*}
x' &= f' \frac{x}{z} \\
y' &= f' \frac{y}{z}
\end{align*}
\]

Note: \( z \) is always negative.

Derived using similar triangles
Pinhole camera

- Common to draw image plane in front of the focal point
- Moving the image plane merely scales the image.

\[
\begin{align*}
x' &= f \frac{x}{z} \\
y' &= f \frac{y}{z}
\end{align*}
\]
Pinhole camera

Is the size of the aperture important?
Cameras & Lenses

Shrinking aperture size
- Rays are mixed up

-Why the aperture cannot be too small?
  - Less light passes through
  - Diffraction effect

Adding lenses!
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Reading:
[FP] Chapters 1 – 3
[HZ] Chapter 6
Cameras & Lenses

- A lens focuses light onto the film
Cameras & Lenses

- A lens focuses light onto the film
  - Rays passing through the center are not deviated
  - All parallel rays converge to one point on a plane located at the focal length $f$
Cameras & Lenses

• A lens focuses light onto the film
  – There is a specific distance at which objects are “in focus”
    [other points project to a “circle of confusion” in the image]
Cameras & Lenses

• Laws of geometric optics
  – Light travels in straight lines in homogeneous medium
  – Reflection upon a surface: incoming ray, surface normal, and reflection are co-planar
  – Refraction: when a ray passes from one medium to another

Snell’s law

\[ n_1 \sin \alpha_1 = n_2 \sin \alpha_2 \]

\( \alpha_1 \) = incident angle
\( \alpha_2 \) = refraction angle
\( n_i \) = index of refraction
Thin Lenses

Snell’s law:
\[ n_1 \sin \alpha_1 = n_2 \sin \alpha_2 \]

Small angles:
\[ n_1 \alpha_1 \approx n_2 \alpha_2 \]
\[ n_2 = n \text{ (lens)} \]
\[ n_1 = 1 \text{ (air)} \]

\[ Z' = f + Z_o \]
\[ f = \frac{R}{2(n-1)} \]

\[ x' = \frac{x}{z} \]
\[ y' = \frac{y}{z} \]
Cameras & Lenses

Issues with lenses: Chromatic Aberration

• Lens has different refractive indices for different wavelengths: causes color fringing

\[ f = \frac{R}{2(n - 1)} \]
Issues with lenses: Spherical aberration

- Rays farther from the optical axis focus closer
Issues with lenses: Radial Distortion

- Deviations are most noticeable for rays that pass through the edge of the lens

No distortion

Pin cushion

Barrel (fisheye lens)

Image magnification decreases with distance from the optical axis
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- The geometry of pinhole cameras
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  - Extrinsic parameters
Relating real-world point to a point on a camera

\[ P = (x, y, z) \rightarrow P' = \left( f \frac{x}{z}, f \frac{y}{z} \right) \]

\[ \mathbb{R}^3 \rightarrow \mathbb{R}^2 \]

f = focal length

c = center of the camera
Relating real-world point to a point on a camera

Is this a linear transformation?

$$P = (x, y, z) \rightarrow P' = (f \frac{x}{z}, f \frac{y}{z})$$

No — division by $z$ is nonlinear!

How to make it linear?
Homogeneous coordinates – a reminder

\[(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\]  
homogeneous image coordinates

\[(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\]  
homogeneous scene coordinates

• Converting \textit{from} homogeneous coordinates

\[
\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w) \\
\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)
\]
Relating a real-world point to a point on the camera

In Cartesian coordinates:

\[ P = (x, y, z) \rightarrow P' = \left( f \frac{x}{z}, f \frac{y}{z} \right) \]

In homogeneous coordinates:

\[
P' = \begin{bmatrix}
    f & x \\
    f & y \\
    z & 0
\end{bmatrix}
= \begin{bmatrix}
    f & 0 & 0 & 0 \\
    0 & f & 0 & 0 \\
    0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z \\
    1
\end{bmatrix}
\]

"Projection matrix"

\[ P' = M \cdot P \quad \mathbb{R}^4 \rightarrow \mathbb{R}^3 \]
Interlude: why does this matter?
Object Recognition (CVPR 2006)
Inserting photographed objects into images (SIGGRAPH 2007)

Original

Created

Slide credit: J. Hayes
Relating a real-world point to a point on the camera

In homogeneous coordinates:

\[
P' = \begin{bmatrix} f & x \\ f & y \\ z \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]

Intrinsic Assumptions
- Unit aspect ratio
- Optical center at (0,0)
- No skew

Extrinsic Assumptions
- No rotation
- Camera at (0,0,0)
Relating a real-world point to a point on the camera

In homogeneous coordinates:

\[ P' = \begin{bmatrix} f & x \\ f & y \\ z \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = K[I \ 0]P \]

**Intrinsic Assumptions**
- Unit aspect ratio
- Optical center at (0,0)
- No skew

**Extrinsic Assumptions**
- No rotation
- Camera at (0,0,0)
Remove assumption: known optical center

**Intrinsic Assumptions**
- Optical center at (0,0)
- Optical center at \((u_0, v_0)\)
- Square pixels
- No skew

\[
P' = K \begin{bmatrix} I & 0 \end{bmatrix} P
\]

\[
w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

**Extrinsic Assumptions**
- No rotation
- Camera at (0,0,0)

---

Slide inspiration: S. Savarese
Remove assumption: square pixels

Intrinsic Assumptions
- Optical center at \((u_0, v_0)\)
- Square pixels
- Rectangular pixels
- No skew

Extrinsic Assumptions
- No rotation
- Camera at \((0,0,0)\)

\[
P' = K \begin{bmatrix} I & 0 \end{bmatrix} P \rightarrow w = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]
Remove assumption: non-skewed pixels

Intrinsic Assumptions
• Optical center at \((u_0, v_0)\)
• Rectangular pixels
• No skew
• Small skew

Extrinsic Assumptions
• No rotation
• Camera at \((0,0,0)\)

\[
P' = K \begin{bmatrix} I & 0 \end{bmatrix} P
\]

\[
\begin{bmatrix}
\alpha & s & u_0 & 0 \\
0 & \beta & v_0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
u \\
v \\
w \\
1
\end{bmatrix}
= \begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

Slide inspiration: S. Savarese
Remove assumption: non-skewed pixels

**Intrinsic Assumptions**
- Optical center at \((u_0, v_0)\)
- Rectangular pixels
- Small skew

**Extrinsic Assumptions**
- No rotation
- Camera at \((0,0,0)\)

\[
P' = K \begin{bmatrix} I & 0 \end{bmatrix} P
\]

\[
w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]
Real world camera: Translate + Rotate
Remove assumption: allow translation

Intrinsic Assumptions
- Optical center at \((u_0, v_0)\)
- Rectangular pixels
- Small skew

Extrinsic Assumptions
- No rotation
- Camera at \((0,0,0) \rightarrow (t_x, t_y, t_z)\)

\[
P' = K \begin{bmatrix} I & \bar{t} \end{bmatrix} P
\]

\[
\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]
Remove assumption: allow rotation

Intrinsic Assumptions
- Optical center at \((u_0, v_0)\)
- Rectangular pixels
- Small skew

Extrinsic Assumptions
- No rotation
- Camera at \((t_x, t_y, t_z)\)

Rotation around the coordinate axes, counter-clockwise

\[
R_x(\alpha) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha \\
\end{bmatrix}
\]
\[
R_y(\beta) = \begin{bmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta \\
\end{bmatrix}
\]
\[
R_z(\gamma) = \begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]
Remove assumption: allow rotation

Intrinsic Assumptions
- Optical center at \((u_0, v_0)\)
- Rectangular pixels
- Small skew

Extrinsic Assumptions
- No rotation
- Camera at \((t_x, t_y, t_z)\)

\[ P' = K [ R \bar{t} ] P \]

\[
\begin{bmatrix}
  u \\
  v \\
  1
\end{bmatrix} =
\begin{bmatrix}
  \alpha & s & u_0 \\
  0 & \beta & v_0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  r_{11} & r_{12} & r_{13} & t_x \\
  r_{21} & r_{22} & r_{23} & t_y \\
  r_{31} & r_{32} & r_{33} & t_z \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
\]
A generic projection matrix

Intrinsic Assumptions
• Optical center at \((u_0, v_0)\)
• Rectangular pixels
• Small skew

Extrinsic Assumptions
• Allow rotation
• Camera at \((t_x, t_y, t_z)\)

\[
P' = K[R \ t]P \quad w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]
A generic projection matrix

Intrinsic Assumptions
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Extrinsic Assumptions
- Allow rotation
- Camera at \((t_x, t_y, t_z)\)

\[
P' = K[R \ t]P \quad \rightarrow \quad w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]

Degrees of freedom??

Slide inspiration: S. Savarese
A generic projection matrix

Intrinsic Assumptions
• Optical center at \((u_0, v_0)\)
• Rectangular pixels
• Small skew

Extrinsic Assumptions
• Allow rotation
• Camera at \((t_x, t_y, t_z)\)

\[
P' = K [R \ t] P \quad \rightarrow \quad \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]

Degrees of freedom??

Slide inspiration: S. Savarese
CS231a: Camera Calibration
estimate all intrinsic and extrinsic parameters

Intrinsic Assumptions
• Optical center at \((u_0, v_0)\)
• Rectangular pixels
• Small skew

Extrinsic Assumptions
• Allow rotation
• Camera at \((t_x, t_y, t_z)\)

\[
P' = K [R \ t] P \quad \rightarrow \quad \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]

Slide inspiration: S. Savarese
Orthographic Projection

- Special case of perspective projection
  - Distance from the COP to the image plane is infinite
  - Also called “parallel projection”
- What’s the projection matrix?

\[
\begin{bmatrix}
  u \\
v \\
w
\end{bmatrix} = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]
Scaled Orthographic Projection

- Special case of perspective projection
  - Object dimensions are small compared to distance to camera

- Also called “weak perspective”
- What’s the projection matrix?

\[
\begin{bmatrix}
  u \\
  v \\
  1
\end{bmatrix} = \begin{bmatrix}
  f & 0 & 0 & 0 \\
  0 & f & 0 & 0 \\
  0 & 0 & 0 & s
\end{bmatrix} \begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
\]
Field of View (Zoom)

From London and Upton
Things to remember

• Vanishing points and vanishing lines

• Pinhole camera model and camera projection matrix $M$
  • Intrinsic parameters
  • Extrinsic parameters

• Homogeneous coordinates
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