Lecture 8: Camera Models

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What we will learn today?

• Pinhole cameras
• Cameras & lenses
• The geometry of pinhole cameras
  • Projection matrix
  • Intrinsic parameters
  • Extrinsic parameters

Reading:
[FP] Chapters 1 – 3
[HZ] Chapter 6
What we will learn today?

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Reading:

[FP] Chapters 1 – 3
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Camera and World Geometry

- How tall is this woman?
- How high is the camera?
- What is the camera rotation?
- What is the focal length of the camera?
- Which ball is closer?
How do we see the world?

• Let’s design a camera
  – Idea 1: put a piece of film in front of an object
  – Do we get a reasonable image?
Pinhole camera

- Add a barrier to block off most of the rays
  - This reduces blurring
  - The opening known as the aperture
Camera obscura: the pre-camera

- Known during classical period in China and Greece (e.g. Mo-Tsi, China, 470BC to 390BC)
Camera Obscura used for Tracing

Lens Based Camera Obscura, 1568
First Photograph

Oldest surviving photograph
- Took 8 hours on pewter plate

Photograph of the first photograph

Joseph Niepce, 1826

Stored at UT Austin

Niepce later teamed up with Daguerre, who eventually created Daguerrotypes

Slide credit: J. Hayes
Dimensionality Reduction Machine (3D to 2D)

3D world

2D image

Point of observation

Figures © Stephen E. Palmer, 2002
Projection can be tricky...

Slide source: Seitz
Projection can be tricky...
Projective Geometry

What is lost?

• Length

Who is taller?

Which is closer?

Slide credit: J. Hayes
Length is not preserved
Projective Geometry

What is lost?

• Length
• Angles
Projective Geometry

What is preserved?

• Straight lines are still straight
Vanishing points and lines

Parallel lines in the world intersect in the image at a “vanishing point”

Slide credit: J. Hayes
Vanishing points and lines

Vanishing line

Vanishing point

Vertical vanishing point (at infinity)

Slide from Efros, Photo from Criminisi
Pinhole camera

\[ P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \]

\[
\begin{align*}
  x' &= f' \frac{x}{z} \\
  y' &= f' \frac{y}{z}
\end{align*}
\]

Note: \( z \) is always negative.

Derived using similar triangles
Pinhole camera

- Common to draw image plane *in front* of the focal point
- Moving the image plane merely scales the image.

\[
\begin{align*}
x' &= f \frac{x}{z} \\
y' &= f \frac{y}{z}
\end{align*}
\]
Pinhole camera

Is the size of the aperture important?
Cameras & Lenses

Shrinking aperture size

- Rays are mixed up

- Why the aperture cannot be too small?
  - Less light passes through
  - Diffraction effect

Adding lenses!
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Reading:
[FP] Chapters 1 – 3
[HZ] Chapter 6
• A lens focuses light onto the film
A lens focuses light onto the film
- Rays passing through the center are not deviated
- All parallel rays converge to one point on a plane located at the focal length $f$
• A lens focuses light onto the film
  – There is a specific distance at which objects are “in focus”
    [other points project to a “circle of confusion” in the image]
Cameras & Lenses

- Laws of geometric optics
  - Light travels in straight lines in homogeneous medium
  - Reflection upon a surface: incoming ray, surface normal, and reflection are co-planar
  - Refraction: when a ray passes from one medium to another

Snell’s law

\[ n_1 \sin \alpha_1 = n_2 \sin \alpha_2 \]

- \( \alpha_1 \) = incident angle
- \( \alpha_2 \) = refraction angle
- \( n_i \) = index of refraction
Thin Lenses

Snell’s law:

\[ n_1 \sin \alpha_1 = n_2 \sin \alpha_2 \]

Small angles:

\[ n_1 \alpha_1 \approx n_2 \alpha_2 \]

\[ n_1 = n \text{ (lens)} \]

\[ n_1 = 1 \text{ (air)} \]

\[ z' = f + z_o \]

\[ f = \frac{R}{2(n - 1)} \]

\[ x' = z' \frac{x}{z} \]

\[ y' = z' \frac{y}{z} \]
Cameras & Lenses
Issues with lenses: Chromatic Aberration

- Lens has different refractive indices for different wavelengths: causes color fringing

\[ f = \frac{R}{2(n-1)} \]
Issues with lenses: Chromatic Aberration

- Rays farther from the optical axis focus closer
Issues with lenses: Chromatic Aberration

- Deviations are most noticeable for rays that pass through the edge of the lens.

- No distortion
- Pin cushion
- Barrel (fisheye lens)

Image magnification decreases with distance from the optical axis.
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  - Extrinsic parameters
Relating real-world point to a point on a camera

\[ \text{image plane} \]
\[ \text{pinhole} \]
\[ \text{virtual image} \]

\[ f = \text{focal length} \]
\[ c = \text{center of the camera} \]

\[ P = (x, y, z) \rightarrow P' = (f \frac{x}{z}, f \frac{y}{z}) \]

\[ \mathbb{R}^3 \rightarrow \mathbb{R}^2 \]
Relating real-world point to a point on a camera

Is this a linear transformation?

\[ P = (x, y, z) \rightarrow P' = \left( f \frac{x}{z}, f \frac{y}{z} \right) \]

No — division by \( z \) is nonlinear!

How to make it linear?
Homogeneous coordinates – a reminder

\[(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\]

homogeneous image coordinates

\[(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\]

homogeneous scene coordinates

• Converting \textit{from} homogeneous coordinates

\[
\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)
\]

\[
\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)
\]
Relating a real-world point to a point on the camera

In Cartesian coordinates:

\[ P = (x, y, z) \rightarrow P' = (f \frac{x}{z}, f \frac{y}{z}) \]

In homogeneous coordinates:

\[
P' = \begin{bmatrix} f & x \\ f & y \\ z \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]

\[ P' = M \cdot P \]

“Projection matrix”

\[ \mathbb{R}^4 \overset{H}{\rightarrow} \mathbb{R}^3 \]
Interlude: why does this matter?
Object Recognition (CVPR 2006)

Slide credit: J. Hayes
Inserting photographed objects into images (SIGGRAPH 2007)

Original

Created

Slide credit: J. Hayes
Relating a real-world point to a point on the camera

In homogeneous coordinates:

\[
P' = \begin{bmatrix} f & x \\ f & y \\ z \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]

Intrinsic Assumptions
- Unit aspect ratio
- Optical center at (0,0)
- No skew

Extrinsic Assumptions
- No rotation
- Camera at (0,0,0)
Relating a real-world point to a point on the camera

In homogeneous coordinates:

\[
P' = \begin{bmatrix} f & x \\ f & y \\ z \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = K[I \ 0]P
\]

Intrinsic Assumptions
- Unit aspect ratio
- Optical center at (0,0)
- No skew

Extrinsic Assumptions
- No rotation
- Camera at (0,0,0)
Remove assumption: known optical center

Intrinsic Assumptions
- Optical center at (0,0)
- Optical center at \((u_0, v_0)\)
- Square pixels
- No skew

Extrinsic Assumptions
- No rotation
- Camera at (0,0,0)

\[ P' = K \begin{bmatrix} I & 0 \end{bmatrix} P \]

\[ w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \]
Remove assumption: square pixels

Intrinsic Assumptions
• Optical center at \((u_0, v_0)\)
• Square pixels
• Rectangular pixels
• No skew

Extrinsic Assumptions
• No rotation
• Camera at \((0,0,0)\)

\[ P' = K \begin{bmatrix} I & 0 \end{bmatrix} P \]

\[
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix} = \begin{bmatrix}
  \alpha & 0 & u_0 & 0 \\
  0 & \beta & v_0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
\]
Remove assumption: non-skewed pixels

Intrinsic Assumptions
- Optical center at \((u_0, v_0)\)
- Rectangular pixels
- No skew
- Small skew

Extrinsic Assumptions
- No rotation
- Camera at \((0,0,0)\)

\[
P' = K \begin{bmatrix} I & 0 \end{bmatrix} P
\]

\[
\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

Slide inspiration: S. Savarese
Remove assumption: non-skewed pixels

Intrinsic Assumptions
• Optical center at \((u_0, v_0)\)
• Rectangular pixels
• Small skew

Extrinsic Assumptions
• No rotation
• Camera at \((0,0,0)\)

\[
P' = K \begin{bmatrix} I & 0 \end{bmatrix} P
\]

Intrinsic parameters

\[
\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]
Real world camera: Translate + Rotate

Slide inspiration: S. Savarese, J. Hayes
Remove assumption: allow translation

Intrinsic Assumptions
• Optical center at \((u_0, v_0)\)
• Rectangular pixels
• Small skew

Extrinsic Assumptions
• No rotation
• Camera at \((0,0,0) \rightarrow (t_x, t_y, t_z)\)

\[
P' = K \begin{bmatrix} I & \overline{t} \end{bmatrix} P
\]

\[
w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]
Remove assumption: allow rotation

Intrinsic Assumptions
• Optical center at \((u_0, v_0)\)
• Rectangular pixels
• Small skew

Extrinsic Assumptions
• No rotation
• Camera at \((t_x, t_y, t_z)\)

Rotation around the coordinate axes, counter-clockwise

\[
R_x(\alpha) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{bmatrix}
\]

\[
R_y(\beta) = \begin{bmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{bmatrix}
\]

\[
R_z(\gamma) = \begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Slide inspiration: S. Savarese
Remove assumption: allow rotation

Intrinsic Assumptions
- Optical center at \((u_0, v_0)\)
- Rectangular pixels
- Small skew

Extrinsic Assumptions
- No rotation
- Camera at \((t_x, t_y, t_z)\)

\[
P' = K \begin{bmatrix} R & \bar{t} \end{bmatrix} P \quad \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]

Slide inspiration: S. Savarese
A generic projection matrix

Intrinsic Assumptions

- Optical center at \((u_0, v_0)\)
- Rectangular pixels
- Small skew

Extrinsic Assumptions

- Allow rotation
- Camera at \((t_x, t_y, t_z)\)

\[
P' = K \begin{bmatrix} R & \bar{t} \end{bmatrix} P \rightarrow \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]

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Extrinsic Assumptions
- Allow rotation
- Camera at \((t_x, t_y, t_z)\)

\[ P' = K[R \ t] P \]

\[ w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \]

Degrees of freedom??

Slide inspiration: S. Savarese
A generic projection matrix

Intrinsic Assumptions
- Optical center at \((u_0, v_0)\)
- Rectangular pixels
- Small skew

Extrinsic Assumptions
- Allow rotation
- Camera at \((t_x, t_y, t_z)\)

\[
P' = K [ R \bar{t} ] P \quad \Rightarrow \quad w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]

Degrees of freedom??

Slide inspiration: S. Savarese
CS231a: Camera Calibration
estimate all intrinsic and extrinsic parameters

Intrinsic Assumptions
• Optical center at \((u_0, v_0)\)
• Rectangular pixels
• Small skew

Extrinsic Assumptions
• Allow rotation
• Camera at \((t_x, t_y, t_z)\)

\[
P' = K [ R \quad \bar{t} ] P \quad \rightarrow \quad w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
\]

Slide inspiration: S. Savarese
Orthographic Projection

- Special case of perspective projection
  - Distance from the COP to the image plane is infinite

- Also called “parallel projection”
- What’s the projection matrix?

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1 \\
\end{bmatrix}
\]

Slide credit: Steve Seitz
Scaled Orthographic Projection

- Special case of perspective projection
  - Object dimensions are small compared to distance to camera
  - Also called “weak perspective”
  - What’s the projection matrix?

\[
\begin{bmatrix}
  u \\
v \\
1
\end{bmatrix} =
\begin{bmatrix}
f & 0 & 0 & 0 \\
0 & f & 0 & 0 \\
0 & 0 & 0 & s \\
1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]
Field of View (Zoom)

From London and Upton
Things to remember

- Vanishing points and vanishing lines
- Pinhole camera model and camera projection matrix $M$
  - Intrinsic parameters
  - Extrinsic parameters
- Homogeneous coordinates

$P' = K \begin{bmatrix} R & t \end{bmatrix} P$

$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Slide inspiration: J. Hayes
What we have learned today?

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[FP] Chapters 1 – 3
[HZ] Chapter 6